THE MINIMAL PAIR PROBLEM FOR HIGHER TYPE OBJECTS

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ABSTRACT

We study recursion in normal functionals of type n+2, as formulated by Kleene, with particular emphasis on the case $n \geq 1$.

In section 0 we recall several definitions of "recursive in F" where F is an object of type n+2: we give first Kleene's definition and then, assuming from there on that F is a normal object, we describe Sacks' definition, a hierarchical definition due to Harrington and a set theoretical definition due, also, to Harrington. In section 1 we show that, under the assumption that F is normal, all these definitions are equivalent.

Let $n \geq 1$. Letting α be the order type of the ordinals having a notation of type n-1 (for recursion in a fixed type n+2 normal object F) we give two versions of the minimal pair problem for all n. In section 2, we use a notion of recursion on α, given by neighbourhood conditions and "F-finite" subsets of α. In section 3, we consider the structure described by Harrington, using only the parts of that structure which correspond to ordinals $\sigma < \alpha$. We give then a construction which gives two (countable) objects of type n+2 forming a minimal pair for objects of type n+1 corresponding to sets of ordinals. We also mention a partial result for two uncountable objects of type n+2 forming a minimal pair. This result is partial because it does not necessarily hold for all n.

Thesis Supervisor: Gerald E. Sacks
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Contents

0. Introduction

In [10] Kleene gave a definition of recursive functionals of finite type. His notion of recursion in a functional of finite type is certainly not the only possible one, not even the only notion which would reasonably generalize what is already known about integers and reals. Nevertheless, Kleene's definition survived. One of the main reasons for this survival is probably the fact that recursion, for Kleene, is actually defined in terms of an inductive definition. This introduces automatically the notions of ordinal and of prewellordering inside Kleene's frame. One can find in [20] an example, among many, of the precious properties of this inductive definition.

In [24], Sacks gave a definition of recursion in normal functionals of finite type (i.e. functionals which among other things have the appropriate equality recursive in them). This definition uses a hierarchy of sets S_σ, for all "possible" σ's. Sacks uses also a lot of induction, including induction on the type. He has shown how to use forcing in the frame of his definition (e.g. in [24] he generalizes a well known result of Kleene about the 1-section of ${}^2E$).

In [8] Harrington gave two apparently different definitions of recursion in normal functionals of finite type. One of the definitions seems to be purely set-theoretical, while the other uses a hierarchy (which is here obviously a hierarchy of sets H_σ, for all suitable σ's, and a lot of model theory. This "universe" is not based on induction on the type.

In the two first sections we will prove that as far as normal objects are concerned, all these definitions are equivalent. In the first section we will only give the first definitions and prove some important lemmas in one of these frames only. The equivalence which we shall prove later will allow us to use these lemmas in all the definitions.

Definition 0.1

$Tp(j)$, the set of objects of type j $(j \in \omega)$ is defined as follows:

$$Tp(0) = \omega$$

$$Tp(n+1) = {}^{Tp(n)}\omega = \text{the set of functions from } Tp(n) \text{ to } \omega$$

Convention

$a^j, b^j, c^j, \ldots$ vary over $Tp(j)$

$k, \ell, m, n,$ vary over ω

A real is a subset of ω. It can also be viewed a function from ω to ω. We will identify the two notions and let ${}^{\omega}\omega = \mathcal{R}$ = the set of all reals. There is an obvious correspondence between $\mathcal{R}$ and Tp(1).

${}^{m}E$ is the equality for objects of type $\ell < m$: we define the function with 2 arguments ${}^{m}E$ as follows

$$ {}^{m}E(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \\ & \text{with } x, y \in \bigcup_{k<m} Tp(k) \end{cases} $$

It is easy to see that we could equivalently use the function with 1 argument, which we shall ambiguously denote also by ${}^{m}E$, defined as follows:

$$ {}^{m}E(a^k) = \begin{cases} 0 & \text{if } (\exists b \in Tp(k-1))(a^k(b) = 0) \\ 1 & \text{otherwise} \\ & \text{with } k < m \end{cases} $$

Clearly, if $m > 1$, ${}^{m}E$ epitomizes the quantification over objects of type ℓ $\ell < m-1$.

An object F of type m is normal iff ${}^{m}E$ is "recursive" in F.

<u>Remarks</u> (for all $k < \omega$)

1) An object of type k can always be viewed as an object of type ℓ, for all $\ell \geq k$

2) A finite sequence of objects of type k can be coded by one single object of type k. The coding operator will be represented by $< \; >$. Conversely an object a of type k can be viewed as coding a finite sequence; in this case $(a)_0, (a)_1, \ldots, (a)_m$ represent the projections.

3) Any subset of $Tp(k)$ can be coded by one single element of $Tp(k+1)$. If $k \geq 1$, then an ω-sequence of elements of $Tp(k)$ can be coded by one single element of $Tp(k)$.

<u>Notation</u>

Fix $n \geq 0$

The objects of type $k \leq n$ are called <u>individuals</u> and the objects of type $k < n$ are called <u>subindividual</u>,

$I = \cup\{Tp(k) \mid k \leq n\}$ = the set of all individuals

$SI = \cup\{Tp(k) \mid k < n\}$ = the set of all subindividuals

The objects of type n+1 are called <u>functions</u> or <u>sets</u> and the objects of type n+2 are called <u>functionals</u>.

F, G, H, ... will denote functionals

f, g, h, ... will denote functions

R, S, T, ... will denote sets

a, b, c, ... will denote individuals

r, s, t, ... will denote subindividuals

<u>Definition 0.2</u> (Kleene-recursive functionals)

<u>Part I: Kleene's schemas</u>

The function Φ on the left hand side (LHS) of the $\simeq$ sign will have n_i arguments of type i for each $i \leq r$ (it is understood that $n_r > 0$) where r is the highest type mentioned as argument of Φ. We shall simultaneously define an index for Φ and Φ itself. $\underline{a}$, $\underline{b}$, $\underline{c}$ stand for appropriate list of arguments; x is an integer; e_1 and e_2 are the indices of the auxiliary functions ψ and χ respectively.

S1 $\Phi(x,\underline{b}) \simeq x + 1$ $\quad\langle 1,\langle n_0,\ldots,n_r\rangle\rangle$

S2 $\Phi(\underline{b}) \simeq k$ $\quad\langle 2,\langle n_0,\ldots,n_r\rangle,k\rangle$

S3 $\Phi(x,\underline{b}) \simeq x$ $\quad\langle 3,\langle n_0,\ldots,n_r\rangle\rangle$

S4 $\Phi(\underline{b}) \simeq \psi(\chi(\underline{b}),\underline{b})$ $\quad\langle 4,\langle n_0,\ldots,n_r\rangle,e_1,e_2\rangle$

S5 $\begin{cases}\Phi(0,\underline{b}) \simeq \psi(\underline{b}) \\ \Phi(x+1,\underline{b}) \simeq \chi(x,\Phi(x,\underline{b}),\underline{b})\end{cases}$ $\quad\langle 5,\langle n_0,\ldots,n_r\rangle,e_1,e_2\rangle$

S6 $\Phi(\underline{a}) \simeq \psi(\underline{c})$ $\quad\langle 6,\langle n_0,\ldots,n_r\rangle,j,k,e_1\rangle$

where $\underline{c}$ is obtained from $\underline{a}$ by moving the k+1 st type j arguments to the front of the list. $(n_j \geq k+1)$

S7 $\Phi(x,a',\underline{b}) \simeq a'(x)$ $\quad\langle 7,\langle n_0,\ldots,n_r\rangle\rangle$

S8 $\Phi(a^j,\underline{b}) \simeq a^j(\lambda c^{j-2}|\chi(a^j,c^{j-2},\underline{b}))$ $\quad\langle 8,\langle n_0,\ldots,n_r\rangle,j,e_2\rangle$

S9 $\Phi(x,\underline{b},\underline{c}) \simeq \{x\}_k(b)$ $\quad\langle 9,\langle n_0,\ldots,n_r\rangle,\langle m_0,\ldots,m_s\rangle\rangle$

where $\{x\}_k$ is the function (partial) Kleene-recursive with index x, s is the maximum type of $\underline{b}$ and $\underline{b}$ has m_i arguments of type i for $i \leq s$.

Part II: An inductive definition

S1 through S9 are the clauses of an inductive definition of a set of triples:

$$K = \{\langle e,\underline{b},\ell\rangle \mid \{e\}_k(\underline{b}) = \ell\}$$

Φ is partial k-recursive via e iff graph (Φ) $= \{\langle \underline{b},\ell\rangle \mid \langle e,\underline{b},\ell\rangle \in K\}$

Φ is k-recursive iff Φ is total and partial k-recursive

Φ is k-primitive recursive iff Φ can be defined using S1-S8, but not S9

Φ is (partial) k-recursive in a type m object F (where F is a total object) iff $\Phi(\underline{b}) \simeq \psi(\underline{b},F)$ with ψ (partial) k-recursive.

Remark

In presence of S9, clause S5 is superfluous.

Part III: Envelopes and sections

$W \subseteq Tp(m)$ is k-semirecursive iff $W = \{b^m \mid \Phi(b^m)$ is defined$\}$ for some Φ, partial k-recursive.

$W \subseteq Tp(m)$ is <u>k-semirecursive in F</u> (where F is a given total object) iff $W = \{b \in Tp(m) \mid \Phi(b,F) \text{ is defined}\}$ for some Φ partial k-recursive.

$R \subseteq Tp(m)$ is <u>k-recursive (in F)</u> iff its characteristic function is k-recursive (in F) iff R and Tp(m)-R are k-semirecursive (in F).

The <u>m-envelope</u> of a total object F is the collection of objects of type m, k-semirecursive in F. The <u>m-section</u> of a total object F is the collection of objects of type m, k-recursive in F. Similar definitions would define the envelopes and sections in the frame of Sacks' and Harrington's definitions.

<u>Definition 0.3</u> (Sacks-recursive functionals)

Fix $n \geq 0$

<u>Part I</u> definition of the hierarchy

Fix F and g

We shall define a hierarchy $\{S_\sigma^{F,g}\}$ of sets, defined by induction on σ; this is designed to define "G is s-recursive in $\langle {}^{n+2}E,F,g\rangle$ via c" by induction on the type.

<u>1) Induction hypothesis</u>

The following predicate: "f is recursive in $\langle {}^{n+1}E,R,a\rangle$

via Gödel number e" has already been defined and we write

$f = \{e\}_s^{<^{n+1}E,R,a>}$

<u>If</u> n = 0, <u>Then</u> f plays the role of a functional for the case "-1" and this definition coincides with Turing's reducibility.

<u>2) Definition of $\{S_\sigma^{F,g}\}$ for all a's (a ε I) simultaneously.</u>

We will define the $S_\sigma^{F,g}$'s in such a way that they contain codes for all ordinals $\tau < \sigma$ and for certain values of F and g.

<u>Stage 0</u>

$S_0^{F,g}$ contains <1,a> for all $a \varepsilon I$ and <1,a,k> whenever ga = k.

<1,a> is an s-index for $S_0^{F,g}$ (for all $a \varepsilon I$) and is associated to the ordinal $|<1,a>|_s = 0$.

<u>Convention</u>

In this thesis we will always assume that a Gödel number is different of 0.

<u>Stage σ+1</u>

$<2^e,a>$ is an s-index for $S_{\sigma+1}^{F,g}$ if

(i) $<2^e,a> \notin S_\sigma^{F,g}$

(ii) $<m,a>$ is an s-index for $S^{F,g}_{\sigma}$ for some $m \varepsilon \omega$ and (iii) there is a function f recursive in $<^{n+1}E, S^{F,g}_{\sigma}, a>$ via the Gödel number e .

(This clause makes sense by induction hypothesis)

The function f of clause (iii) is denoted by $\lambda b \mid \{e\}^{G}_{s}(b)$ where $G = <^{n+1}E, S^{F,g}_{\sigma}, a>$ and $\{e\}^{G}_{s}$ is the e^{th} function s-recursive in G.

<u>If</u> $<2^e,a>$ is an s-index for $S^{F,g}_{\sigma+1}$

<u>Then</u> we associate to it the ordinal $|<2^e,a>|_s = \sigma+1$

Let $S^{F,g}_{\sigma+1} = S^{F,g}_{\sigma} \cup \{<2^e,a> \mid <2^e,a>$ is an s-index for $S^{F,g}_{\sigma+1}\}$

$\cup \{<3^e,a,b,k> \mid <2^e,a>$ is an s-index for $S^{F,g}_{\sigma+1}$ & $\{e\}^{G}_{s}(b) = k$ & $G = <^{n+1}E, S^{F,g}_{\sigma}, a>\}$

$\cup \{<5^e,a,k> \mid <2^e,a>$ is an s-index for $S^{F,g}_{\sigma+1}$ & $F(\{e\}^{G}_{s}) = m$ & $G = <^{n+1}E, S^{F,g}_{\sigma}, a>\}$

<u>Stage λ (λ limit)</u>

$<7^e,a>$ is an s-index for $S^{F,g}_{\lambda}$ if $<2^e,a>$ is an s-index for some $\delta+1 < \lambda$ and $\lambda b \mid \{e\}^{G}_{s}(b)$

(with $G = \langle {}^{n+1}E, S_{\delta}^{F,g}, a\rangle$) is the characteristic function of a set T of s-indices such that

$$\lambda = \sup \{|b|_s \mid b \in T\}$$

<u>If</u> b is an s-index for $S_{\lambda}^{F,g}$, we associate to it the ordinal $|b|_s = \lambda$.

Let $S_{\lambda}^{F,g} = \underset{\delta<\lambda}{\ } S_{\delta}^{F,g} \quad \{b \mid b \text{ is an s-index for } S_{\lambda}^{F,g}\}$.

3) Let $\kappa^{F,g} = \ell.u.b.\{\sigma \mid \sigma$ is the image by $|\ |_s$ of an s-index$\}$ and $S_{\infty}^{F,g} = \ \{S_{\sigma}^{F,g} \mid \sigma < \kappa^{F,g}\}$

<u>Part II:</u> <u>definition of $G \leq_s \langle {}^{n+2}E, F\rangle$</u>

1) a function f is said to be <u>s-recursive in</u> $G = \langle {}^{n+2}E, F, g, a\rangle$ with Gödel number e iff

$\langle 2^e, a\rangle$ is an s-index, say $|\langle 2^e, a\rangle|_s = \sigma + 1$

and $f = \{e\}_s^{G'}$ where $G' = \langle {}^{n+1}E, S_{\sigma}^{F,g}, a\rangle$.

f is denoted by $\{e\}_s^G$.

f is said to be <u>s-recursive</u> in $G = \langle {}^{n+2}E, F, g\rangle$ iff there is an integer e such that f is s-recursive in $\langle {}^{n+2}E, F, g, 0\rangle$ via e iff $f = \{e\}_s^G$ for some e.

G is said to be s-recursive in $H = \langle {}^{n+2}E, F, g\rangle$ via e, and this is written $G = \{e\}_s^H$, iff $G(h) = \{e\}_s^{H*h}(0)$ for all functions h (with $H*h = \langle {}^{n+2}E, F, g*h, 0\rangle$ where $g*h$ is the function coding the pair (g,h)).

G is said to be s-recursive in $\langle {}^{n+2}E, F\rangle$ via e, and this is written $G = \{e\}_s^{\langle {}^{n+2}E, F\rangle}$ iff $G = \{e\}_s^H$ with $H = \langle {}^{n+2}E, F, 0\rangle$.

Part III

Let R be a set.

$R \subseteq I$ is s-semirecursive in $\langle {}^{n+2}E, F\rangle$ iff $(\exists e \in \omega)[R = \{a \mid \langle 2^e, a\rangle \text{ is an s-index}\}]$

$R \subseteq I$ is s-recursive in $\langle {}^{n+2}E, F\rangle = G$ iff its characteristic function is s-recursive in G iff R and I-R are both semirecursive in G iff there is an integer e such that $R = \{e\}_s^G$ i.e. $\langle 2^e, 0\rangle$ is an s-index, say $|\langle 2^e, 0\rangle|_s = \sigma + 1$ and $R = \{e\}_s^{\langle {}^{n+1}E, S_\sigma^{F,g}\rangle}$.

Envelopes and sections are defined as in definition 0.2.

Definition 0.4 Harrington's recursive universe.

Fix $n \geq 0$

Let $\mathcal{I} = \langle I, \varepsilon, \ldots \rangle$ be a structure such that (1) $\mathcal{I}$ is closed under coding and decoding of finite sequences and (2) $\omega \subseteq I$ (e.g. let $\mathcal{I}$ be $\langle R_{\omega+n}, \varepsilon \rangle$)

${}^{n+2}E$ is the equality for subsets of I (or for elements of ${}^{I}\omega$) and thus introducts the quantifiers for elements of I.

Let $F : {}^{I}\omega \to \omega$ be, from now on, a fixed functional.

<u>Convention</u>: as we want to work with F and the equality we may as well assume that F is $\langle F', {}^{n+2}E \rangle$ where F' is a fixed functional.

<u>Part I: definition of a jump</u>

Let $X \subseteq Tp(n) = I$

Let $\{e\}_p^X$ be the e^{th} function (via some natural Gödel numbering of the formulas) which is first order definable over $\langle \mathcal{I}, X \rangle$

Let $W_e^X = \{a \in I \mid \{e\}_p^X(a) = 0\}$

Let jX (the "jump" of X) be defined as follows:

$jX = \{\langle e,a,0 \rangle \mid a \varepsilon W_e^X\} \cup \{\langle e,a,y+1 \rangle \mid F(\{e\}_p^{\langle X,a \rangle}) = y\}.$

Note that $jX \subseteq \omega \times I \times \omega$

j is defined in such a way that jX tells us which subsets of I can be defined from X in a first order way and j gives for each such subset of I its image by the fixed functional F.

Part II: the h-hierarchy

We define now, inductively and simultaneously

1) a subset $\mathcal{O}^F \subseteq \omega \times I$

2) a function $| \ |^F : \mathcal{O}^F \to On$

3) for each ordinal σ in the range of $| \ |^F$, a subset $H^F_\sigma \subseteq I$

1) for all $a \in I$ $\qquad \langle 1,a\rangle \in \mathcal{O}^F$

$$|\langle 1,a\rangle|^F = 0$$

$$H^F = \emptyset$$

2) If $\langle e,a\rangle \in \mathcal{O}^F$ (say $|\langle e,a\rangle|^F = \sigma$)

Then $\langle 2^e,a\rangle \in \mathcal{O}^F$

$$|\langle 2^e,a\rangle|^F = \sigma + 1$$

$$H^F_{\sigma+1} = j(H^F_\sigma)$$

3) If $\begin{bmatrix} \langle m,a\rangle \in \mathcal{O}^F \quad (\text{say } |\langle m,a\rangle|^F = \sigma) \\ W_e^{\langle H^F_\sigma, a\rangle} \subseteq \mathcal{O}^F \end{bmatrix}$

Then $\langle 3^m \cdot 5^e,a\rangle \in \mathcal{O}^F$

$|\langle 3^m \cdot 5^e,a\rangle|^F = \lambda$ = first limit ordinal strictly bigger than any ordinal having a notation in the set

$\{\langle m,a\rangle\} \cup W_e^{\langle H_\sigma^F, a\rangle} = \mu x \in On\ [x$ is a limit ordinal & $x > \sigma$

& $(\forall b \in W_e^{\langle H_\sigma^F, a\rangle})(x > |b|^F)]$

$$H_\lambda^F = \{\langle b,c\rangle \in I \mid b \in \mathcal{O}^F \ \&\ |b|^F < \lambda \ \&\ c \in H^F_{|b|^F}\}$$

[When there is no risk of confusion, we shall sometimes write $W_e^{H_\sigma^F}$ instead of $W_e^{\langle H_\sigma^F, a\rangle}$]

<u>Remarks</u>

i) We defined $\mathcal{O}^F$ as a subset of $\omega \times I$, but by a previous remark this can be identified with a subset of I

ii) The definition of $|\langle 3^m \cdot 5^e, a\rangle|^F$ in clause 3) makes sense because by <u>hypothesis</u> each member of $\{\langle m,a\rangle\} \cup W_e^{\langle H_\sigma^F, a\rangle}$ is already known to be a notation for an ordinal

iii) Remember that we decided that 0 is not a Gödel number.

<u>Part III definition of $G \leq_h F$</u>

The function f is said to be h-recursive iff there is an integer e such that $f \simeq \{e\}^F_h$ where $\{e\}^F_h(a) \simeq x$ is defined as follows:

$$\{e\}^F_h(a) \simeq x \text{ iff } e = \langle e_0, e_1\rangle \ \&\ \langle e_0, a\rangle \in \mathcal{O}^F$$

(say $|\langle e_0, a\rangle|^F = \sigma$) & $\{e_1\}_p^{H^F_\sigma}(a) = x$ & $x \in \omega$.

i.e. $\{e\}^F_h(a) \simeq x$ means

1) $e = \langle e_0, e_1\rangle$

2) $\langle e_0, a\rangle \in \mathcal{O}^F$ (say $|\langle e_0, a\rangle|^F = \sigma$)

3) $\langle I, H^F_\sigma\rangle$ says that the e_1 th function first order definable over $\langle I, H^F_\sigma\rangle$ takes the value x for the argument a & x is an integer.

<u>Notation</u>

$\{e\}^F_h(a)\downarrow$ means $\{e\}^F_h(a) = x$ for some $x \in \omega$

$\{e\}^F_h(a)\uparrow$ otherwise.

Let $G \in Tp(n+2)$

G is said to be <u>recursive in F</u> via Gödel number e iff $(\forall f \in Tp(n+1))(G(f) \simeq \{e\}_h^{\langle F, f\rangle}(0))$

A partial function $\Phi: Tp(n+2) \to \omega$ is <u>recursive in</u> F iff for some integer e and for all $G \in Tp(n+2)$, $\Phi(G) \simeq \{e\}_h^{\langle F, G\rangle}(0)$.

A subset R of $Tp(n+2)$ is <u>semirecursive in</u> F iff it is the domain of a partial function recursive in F.

A subset R of $Tp(n+2)$ is <u>recursive in</u> F iff its characteristic function is recursive in F iff R and $Tp(n+2)-R$ are semirecursive in F.

We would like now to prove a few lemmas in Harrington's frame. When we are sure that it will not lead to confusing statements we will drop the subscript h or the superscript F. In this section, $\{e\}^G$ should thus be read as $\{e\}_h^{\langle F,G\rangle}$

We will also from now on adopt an anthropomorphic attitude: e.g. ordinals will be able to see, realize, say, tell us, ..., whatever can be deduced from them, or better from their level of the hierarchy $\{H_\sigma\}$. We will prove the following lemma to show how elegant this method can be.

<u>Lemma 0.1</u>

There is a recursive function $f : \omega \to \omega$ such that for every integer e and individual a, <u>if</u>

$\{e\}^F(a) \simeq x \ \& \ \langle x,a\rangle \in \mathcal{O}^F$ <u>then</u>

(i) $\langle f(e),a\rangle \in \mathcal{O}^F$

(ii) $|\langle f(e),a\rangle| \geqslant |\langle x,a\rangle|$

Proof

Assume $\{e\}^F(a) \simeq x$ & $\langle x,a\rangle \in \mathcal{O}^F$

$\therefore$ (by definition) $e = \langle e_0,e_1\rangle$ & $\langle e_0,a\rangle \in \mathcal{O}^F$

(say $|\langle e_0,a\rangle| = \sigma$)

Let $A = \{b \in I \mid \langle I,H_\sigma\rangle \models \phi_{e_1}(b,a,\ldots)\}$ where

ϕ_{e_1} is the graph of $\{e_1\}_p^{H_\sigma}$

Note that 1) $x \in A$ (by definition of $\{e\}^F(a) \simeq x$)

2) $x \in \omega$ (by definition of $\{e\}^F(a) \simeq x$)

3) x is the smallest integer y such that $y \in A$

If $A \subseteq \mathcal{O}^F$

then $\langle 3^{e_0}\cdot 5^{e_1},a\rangle \in \mathcal{O}^F$, say $|\langle 3^{e_0}\cdot 5^{e_1},a\rangle| = \lambda$

$\therefore$ $\lambda > |u|$ for all $u \in A$, hence $\lambda > |x|$

But this does not give us any information about $\langle x,a\rangle$ and A might have as element some $b \in I - \mathcal{O}^F$. To solve this problem we define A' as follows:

$A' = \{\langle y,a\rangle \mid y$ is the smallest integer in $A\}$

$\therefore$ $A' = \{\langle x,a\rangle\}$ (a is fixed of course)

By hypothesis $A' \subseteq \mathcal{O}^F$.

As $e = \langle e_0, e_1 \rangle$ gives us all the information needed to compute A, and as we can identify the integers among the element of I, we can find effectively a Gödel number e_2 such that

$$A' = \{b \in I \mid \langle I, H_\sigma \rangle \quad \phi_{e_2}(b,a)\}$$

$$\therefore \quad \langle 3^{e_0} \cdot 5^{e_2}, a \rangle \in \mathcal{O}^F$$

and $|\langle 3^{e_0} \cdot 5^{e_2}, a \rangle| = \lambda > |\langle x, a \rangle|$

Let $f(e) = 3^{e_0} \cdot 5^{e_2}$ ⊣

Definition 0.4

1) derived notation

We define a function $f_0 : I \to I$ as follows:

$$f_0(a) = \begin{cases} a & \text{if } a = \langle 1, b \rangle \text{ for some } b \in I \\ \langle n, b \rangle & \text{if } a = \langle 2^n, b \rangle \quad (n \in \omega) \\ \langle n, b \rangle & \text{if } a = \langle 3^n 5^e, b \rangle \quad (n, e \in \omega) \\ \text{undefined otherwise.} \end{cases}$$

$f_0(a)$, if defined, is called the derived notation of a. A derived notation $f_0(a)$ is not necessarily a member of $\mathcal{O}^F$, but an object which might be able to say that a is a member of $\mathcal{O}^F$.

2) <u>If</u> $a = \langle 3^n \cdot 5^e, b\rangle$ $(n, e \in \omega)$ and

i) e is a Gödel number

ii) $\langle n, b\rangle \in \mathcal{O}^F$ (say $|\langle n, b\rangle| = \tau$)

<u>Then</u> we call the set $\{i \in I \mid \langle I, H_\tau\rangle \models \phi_e(i, a)\}$ the <u>set defined by a</u>.

3) <u>If</u> $b \in \mathcal{O}^F$ and $b = \langle n, b'\rangle$ for some $n \in \omega$, $b' \in I$

<u>Then</u> b is called a <u>notation from b'</u> for $|b|$ and $|b|$ is said to have an index from b'.

<u>Definition 0.5</u>

1) $\kappa^F = \ell.u.b. \{\sigma \in On \mid \sigma$ has an index from some individual$\}$

2) For each ordinal $\sigma < \kappa^F$, we define the set

$\mathcal{O}^F_\sigma = \{c \in I \mid c \in \mathcal{O}^F \ \&\ |c| < \sigma\}$

3) It is convenient to adopt the following notation: if $a \notin \mathcal{O}^F$, we let $|a| = \infty$.

<u>Lemma 0.2</u>

<u>If</u> $a \in \mathcal{O}^F \ \&\ b \in \mathcal{O}^F \ \&\ |a| < |b|$

<u>Then</u>

i) $H^F_{|a|}$ is primitive recursive in $\langle H^F_{|b|}, a, b\rangle$

ii) $\mathcal{O}^F_{|a|}$ is primitive recursive in $\langle H^F_{|b|}, a, b\rangle$

(Remember that "primitive recursive in X" means "first order definable over $\langle \mathcal{I}, X\rangle$")

Proof (by induction on $|a|$)

Case 0: $|a| = 0$, $|b|$ is any ordinal $\sigma < \kappa^F$

Then $H^F_0 = \mathcal{O}^F_0 = \emptyset$

and $\emptyset = \{x \in I \mid \mathcal{I} \quad x \neq x\}$

Case 1 $|a| = \sigma+1 \leq |b| < \kappa^F$

Then a is of the form $\langle 2^e, a'\rangle$ and $\langle e, a'\rangle$ is a notation for σ

$\therefore \quad |\langle e, a'\rangle| < |b|$

$\therefore \quad H_{|\langle e,a'\rangle|}$ is primitive recursive in $\langle H_{|b|}, \langle e,a'\rangle, b\rangle$

and as e can obviously be recovered from 2^e we have:

H_σ is primitive recursive in $\langle H_{|b|}, \langle 2^e, a'\rangle, b\rangle$

and also $\mathcal{O}^F_\sigma$ is primitive recursive in $\langle H_{|b|}, \langle 2^e, a'\rangle, b\rangle$

But $\langle k,x\rangle \in \mathcal{O}^F_{\sigma+1} \iff [k=1] \vee [(\exists c \in \omega)(k = 2^c \;\&$

$\langle c,x\rangle \in \mathcal{O}^F_\sigma)] \vee [(\exists c \in \omega)(\exists m \in \omega)(k = 3^m \cdot 5^c) \;\&\; \langle m,x\rangle \in \mathcal{O}^F_\sigma$

$\& \;(\forall y)(y \in W_e^{H_{|\langle m,x\rangle|}} \Rightarrow y \in \mathcal{O}^F_\sigma)]$.

By hypothesis a is a notation for $\sigma+1$, hence $f_0 a$ gives us effectively a notation for σ and membership of $\mathcal{O}^F_\sigma$ is primitive recursive in $\langle H_{|b|},a,b\rangle$. But then $\langle m,x\rangle$ is a notation for some ordinal $\tau < \sigma$ and $y \varepsilon W_e^{H_\tau}$ is a relation primitive recursive in $\langle H_{|b|},a,b\rangle$.

We can now give a definition for $H_{\sigma+1}$:

$$\langle k,y,m\rangle \varepsilon H_{\sigma+1} \iff [m = 0 \;\&\; y \varepsilon W_k^{H_\sigma}] \vee [m > 0$$
$$\& \; F(\{k\}_p^{\langle H_\sigma,y\rangle}) = m - 1]$$

As before $f_0 a$ is an index for σ, we have H_σ primitive recursive in $\langle H_{|b|},a,b\rangle$. Thus we have a first order definition for H_σ over $\langle I,\langle H_{|b|},a,b\rangle\rangle$. Using this definition wherever needed, we obtain a first order definition of $H_{\sigma+1}$ over $\langle I,\langle H_{|b|},a,b\rangle\rangle$

Case 2 $|a| = \lambda$, with λ a limit ordinal

Then a is of the form $\langle 3^m \cdot 5^e,a'\rangle$ and $f_0 a = \langle m,a'\rangle$ is a notation for an ordinal $\tau < \lambda$, such that

$$W_e^{\langle H_\tau,a\rangle} \subseteq \mathcal{O}^F_\lambda$$

We can then define $\mathcal{O}^F_\lambda$ as follows:

$$c \in \mathcal{O}^F_\lambda \iff (\exists k \in \omega)(\exists x)(x \in W_e^{<H_\tau, a>} \cup \{f_0 a\})(c \in \mathcal{O}^F_{|<f_1(k,(x)_0),(x)_1>|})$$

where $f_1 : \omega \times \omega \to \omega$ is defined as follows:

$$\begin{cases} f_1(0,d) = d \\ f_1(\ell+1,d) = 2^{f_1(\ell,d)} \end{cases}$$

This definition of $\mathcal{O}^F_\lambda$ is simply using the fact that, by hypothesis, $W_e^{<H_\tau, a>} \subseteq \mathcal{O}^F_\lambda$. The function f_1 is such that if $<m,d>$ is a notation for σ, then $<f_1(k,m),d>$ is a notation for $\sigma+k$. We can thus reach all "finite successors" of ordinals having notations in $W_e^{<H_\tau, a>} \cup \{f_0 a\}$. λ is precisely the first limit ordinal bigger than all these ordinals.

It is then easy to define H_λ as follows:

$$<b,c> \in H_\lambda \iff b \in \mathcal{O}^F_\lambda \ \& \ c \in H_{|b|}$$

⊣

Remark

In the proof of lemma 0.2 we had to solve simultaneously the $\mathcal{O}^F_\sigma$ and the H_σ parts. This will happen again and again ...

Corollary 0.3

If b and c are notations for ordinals and a is an individual

Then the relations "b is a notation from a" and "b and c are notations from the same individual" are primitive recursive.

(Corollary 0.3 would be false without the "if" part!)

We will now give an "anthropomorphic" version of lemma 0.2.

Fact 0.4 (Fact G)

If $a \in \mathcal{O}^F$ and $|a| \leq \tau < \kappa^F$
Then H_τ^F can recognize it as such

(i.e.: if $\tau < \kappa^F$ (thus we have an index for τ), and $|a| \leq \tau$, then we can build H_τ and check in the structure $\langle I, H_\tau \rangle$ whether a is, or is not, a member of $\mathcal{O}^F_\tau$.)

Note that if H_τ does not recognize a (as a member of $\mathcal{O}^F_\tau$), then either a is a notation for an ordinal $\sigma > \tau$ or $a \notin \mathcal{O}^F$; but H_τ does not tell us which is the case. (Note also that, when we use fact G, it is important to get effectively an index for τ!)

Notation

To make the proof of the following theorems easier to read (and write), we will sometimes use -*- as coding operator: <k,a*b> should thus be read <k,<a,b>>.

We want now to show that it is possible to compare notations ("stage comparison theorem"). This theorem, due to Gandy, implies the existence of a selection operators for integers:

"there is a recursive function $\lambda e \mid e'$ $(e, e' \in \omega)$ such that, if T is a non empty subset of ω, semirecursive in F with Gödel number e, then $\{e'\}^F(0)$ is defined and belongs to T."

We will prove the following result with many details because we want to use this proof to prove further results.

Theorem 0.5 (Gandy)

There is a recursive function $f : I \times I \to \omega$ such that if $a \in \mathcal{O}^F$ or $b \in \mathcal{O}^F$, then $f(a,b)$ is defined, say $f(a,b) = k$, and the following hold

(i) $\langle k, a*b \rangle \in \mathcal{O}^F$

(ii) $|\langle k, a*b \rangle| \geq \min\{|a|, |b|\}$

Proof

The proof is by induction on the min:

Induction hypothesis: $(\forall a', b' \in I)$

$\underline{\text{If}}$ $\min \{|a'|, |b'|\} < \min \{|a|, |b|\}$

$\underline{\text{Then}}$ $(\exists k' \in \omega)(\langle k', a'*b'\rangle \in \mathcal{O}^F$ &

$|\langle k', a'*b'\rangle| > \min \{|a'|, |b'|\})$

This k' is $f(a',b')$ (and thus can be found effectively).

$\underline{\text{Case 1}}$ $f_0 a = a$ or $f_0 b = b$

(i.e. $|a| = 0$ or $|b| = 0$)

let $f(a,b) = 1$

$\underline{\text{Case 2}}$ $a = \langle 2^{m_0}, a'\rangle$ and $b = \langle 2^{m_1}, b'\rangle$

We assume that at least one of a and b in $\mathcal{O}^F$ (otherwise there is nothing to prove)

$\therefore$ $f_0 a \in \mathcal{O}^F$ or $f_0 b \in \mathcal{O}^F$

But $\min \{|f_0 a|, |f_0 b|\} < \min \{|a|, |b|\}$

$\therefore$ by induction hypothesis, we can find effectively a notation $\langle m, c\rangle$ for an ordinal σ such that $\sigma \geq \min \{|f_0 a|, |f_0 b|\}$, with $m = f(f_0 a, f_0 b)$ and $c = f_0 a * f_0 b$.

Let m' be such that m' carries all the information needed to go from $a*b$ to c, and also the information contained in m.

Let $f(a,b) = 2^{m'}$.

Case 3 assume $a = <2^{e_0}, a'>$ and $b = <3^m \cdot 5^{e_1}, b'>$

assume also $a \in \mathcal{O}^F$ or $b \in \mathcal{O}^F$

$f_0 a \in \mathcal{O}^F$ or $f_0 b \in \mathcal{O}^F$

$\therefore$ by induction hypothesis, we get effectively

$d = <p, f_0 a * f_0 b> \in \mathcal{O}^F$

We would like to get hold of an ordinal τ, big enough to recognize (as in fact G) whether $(f_0 a \in \mathcal{O}^F)$ or $(f_0 b \in \mathcal{O}^F)$ holds [Remember that it must be the case for at least one of them] and such that τ has an index directly from $a*b$. $|d|$ can only try to do the first part of this, but from d we can get effectively $p' \in \omega$ such that p' carries all the information needed to go from $a*b$ to $f_0 a * f_0 b$, and the information contained in p. Let $d' = <p', a*b>$ and $\tau = |d'|$.

We do not know whether τ is $\geq \min\{|a|,|b|\}$, but we have an index for τ, hence we can build H_τ and we can compute from it. Hence we can recognize whether $(\tau \geq |f_0 a|)$ or $(\tau \geq |f_0 b|)$ holds.

Case 3.1

H_τ "says" $\tau \geq |f_0 a|$

Then a is in $\mathcal{O}^F$: let $f(a,b) = 2^{p'}$

$\therefore$ $|<2^{p'}, a*b>| = \tau+1 \geq |f_0 a| + 1 = |a|$.

Case 3.2

This is not the case

$\therefore$ H_τ "says" $\tau \geq |f_0 b|$

$\therefore$ τ is big enough to compute the set B defined by b.

Does this mean that $b \in \mathcal{O}^F$ & $|b| \leq \tau$?

Everything would be fine if

$B = \{x \in I \mid \langle I, H_{|f_0 b|}\rangle \models \phi_{e_1}(x,b')\}$ was a subset of $\mathcal{O}^F$.

But $|f_0 b|$ might be too small to show this and anyway, this might be false. Luckily, as τ is big enough (and as we got effectively an index for τ) we can compute effectively from e_1 a Gödel number e_2 which carries the information needed to go from τ to $|f_0 b|$ (i.e. from the index we got for τ to $f_0 b$) and the information contained in e_1, such that

$$B = \{x \in I \mid \langle I, H_\tau\rangle \models \phi_{e_2}(x,b')\}$$

We have now a definition of B over $\langle I, H_\tau\rangle$ from b', thus we can find effectively a Gödel number e_3 such that $B = \{x \in I \mid \langle I, H_\tau\rangle \models \phi_{e_3}(x, a*b)\}$.

We will not compare each member of B with $f_0 a$: it could be the case that $f_0 a$ and $f_0 b$ are both in $\mathcal{O}^F$, but that $|f_0 a| > \tau \geq |f_0 b|$, with $a \in \mathcal{O}^F$ and $b \notin \mathcal{O}^F$.

Let $g : I \to I$ be defined as follows:

$$g(c) = \begin{cases} 1 & \text{if } c \notin B \\ \langle k, f_0 a * c \rangle & \text{if } c \in B \\ & \text{with } k = f(f_0 a, c) \end{cases}$$

(By induction hypothesis k is well defined)

Clearly, g is recursive in B, thus in H_τ (together with a and b).

Note that $g[B] \subseteq \mathcal{O}^F$

We can now find effectively a Gödel number e_4 such that $g[B] = \{x \in I \mid \langle I, H_\tau \rangle \models \phi_{e_4}(x, a*b)\}$

Let then $q = 3^{p'} \cdot 5^{e_4}$ (remember that $\langle p', a*b \rangle$ is an index for τ)

Then $\langle q, a*b \rangle \in \mathcal{O}^F$ and

$\tau' = |\langle q, a*b \rangle| = 1^{st}$ limit ordinal strictly bigger than all members of $\{|gc| \mid c \in B\} \cup \{\tau\}$

$\therefore \quad \tau' > \tau \geq |f_0 b|$

and τ' is big enough to check that $B \subseteq \mathcal{O}^F$ or, if this is not the case, $\tau' \geq |f_0 a|$

Let $f(a,b) = 2^q$

Then $<2^q, a*b>$ is a notation from $a*b$ for an ordinal $\sigma \geq \min\{|a|,|b|\}$.

Case 4 assume $a = <3^{m_0} \cdot 5^{3_0}, a'>$ and $b = <3^{m_1} \cdot 5^{e_1}, b'>$ and assume that at least one of a or b is in $\mathcal{O}^F$.

(1) By induction we know that $f(f_0a, f_0b)$ is defined, say $f(f_0a, f_0b) = k$ and that $<k, f_0a*f_0b> \in \mathcal{O}^F$

Let $\tau = |<k, f_0a*f_0b>| = |<k', a*b>|$ where k' carries as usually the information contained in k and the instructions needed to go from $a*b$ to f_0a*f_0b.

Remark: we have obtained effectively an index for τ and thus, we can use fact G without problem.

(2) τ knows which one (of f_0a and f_0b) is in $\mathcal{O}^F_{\tau+1}$ (by fact G), if one of them is in $\mathcal{O}^F$. But by hypothesis we know that this must be the case.

Claim 1

τ can recognize at least one of f_0a, f_0b as a notation for a smaller (or equal) ordinal.

Proof of the claim: by definition of $k = f(f_0a, f_0b)$

claim (1)

Assume τ recognizes f_0a. Then τ is big enough to let us define correctly the set A defined by a (over $<I,H_\tau>$).

If $A \subseteq O^F$, we are done; but this might be false. So look at every member a_i of A and compare it to f_0b. We know that either $A \subseteq O^F$ ($\therefore a_i \in O^F$ for all i) or $b \in O^F$ ($\therefore f_0b \in O^F$). [It is of course possible to have $A \subseteq O^F$ and $f_0b \in O^F$]. Thus for all pairs $(a_i, f_0b) \in A \times \{f_0b\}$ we have: at least one of a_i, $f_0b \in O^F$ and hence min $\{|a_i|, |f_0b|\}$ is defined and strictly smaller than min $\{|a|, |b|\}$.

$\therefore$ by induction hypothesis, we will be able to use the function f to compare each $a_i \in A$ and f_0b.

We get thus a set of ordinals $T = \{\tau_{a_i} \mid a_i \in A\}$ corresponding to the set $\tilde{T}$ of notations for ordinals which we get effectively by induction hypothesis.

Clearly we have a first order definition, say ϕ_{e_2}, of $\tilde{T}$ over $<I,H_\tau>$ from a*b.

Let τ' be the least limit ordinal strictly bigger than all members of $T \cup \{\tau\}$. It is also obvious that we get τ' effectively and that $<3^{k'} \cdot 5^{e_2}, a*b>$ is a notation for τ'.

Claim 2

τ' must either recognize f_0b as a member of $\mathcal{O}^F$ (while τ did not do so) or recognize A as a subset of $\mathcal{O}^F$. τ' must be such that for one of the two facts mentioned, no ordinal σ obtained effectively, such that, $\sigma < \tau'$ can give the same conclusion as τ' (i.e.: if there is still something new to mention, τ' mention such a thing.)

Proof of the claim

$\tau' \geq \tau$ (and we have an index for τ')

$\therefore$ τ' recognizes f_0a as a notation.

Assume τ' does not tell us that $A \subseteq \mathcal{O}^F$. By hypothesis $A \subseteq \mathcal{O}^F$ or $b \in \mathcal{O}^F$. If τ' does not say that $A \subseteq \mathcal{O}^F$, then for some $x \in A$, we have: either $x \notin \mathcal{O}^F$ or $x \in \mathcal{O}^F$ & $|x| > \tau'$.

$\therefore$ when we compare x and f_0b, as we know that at least one of x and f_0b is a member of $\mathcal{O}^F$ <u>and</u> that $\min\{|x|, |f_0b|\} < \min\{|a|, |b|\}$, we have (by induction hypothesis):

1) $\langle f(x, f_0b), x * f_0b\rangle \in \mathcal{O}^F$

2) $|\langle f(x, f_0b), x * f_0b\rangle| = \alpha < \tau'$

3) $\alpha \geq \min\{|x|, |f_0b|\}$

Thus if τ' does not identify A as a subset of $\mathcal{O}^F$, we know that τ' must recognize $f_0 b$ as a notation for a smaller ordinal.

τ' is the first limit ordinal strictly bigger than all members of $T \cup \{\tau\}$, thus (by definition of $\mathcal{O}^F$) the first ordinal strictly bigger than all the members of $T \cup \{\tau\}$ for which we can expect to get (effectively) a notation, <u>whatever the sets A and $\tilde{T}$ are</u>.

claim (2)

(3) <u>If</u> τ' says "$A \subseteq \mathcal{O}^F$ holds"

<u>Then</u> we are done

but <u>if</u> τ' does not tell us that $A \subseteq \mathcal{O}^F$

<u>Then</u> τ' says that $f_0 b \in \mathcal{O}^F$

$\therefore$ τ' is big enough to let us define correctly the set B defined by b over $<\mathcal{I}, H_{\tau'}>$

Clearly we get then effectively (remember: we have an index for τ') a definition ϕ_{e_3} of B and a definition ϕ_{e_4} of A over $<\mathcal{I}, H_{\tau'}>$ from a*b.

By induction hypothesis we can compare, without problem, all the members of A with all the members of B (remember that at least one of A and B is a subset of $\mathcal{O}^F$)

This will give us effectively (notation for) ordinals $\tau_{a_i b_j}$ for all $(a_i, b_j) \varepsilon A \times B$. We have notations for these ordinals, for τ and for τ': hence we can define a formula satisfied only by these notations. All this can be done <u>effectively</u> because we took <u>all possible</u> pairs $(a_i, b_j) \varepsilon A \times B$. Thus we get a notation from $a*b$, for τ'', the first limit ordinal strictly bigger than τ, τ' and all the $\tau_{a_i b_j}$'s [Note that we got a notation for τ' via a similar argument, taking <u>all possible</u> pairs $(a_i, f_0 b) \varepsilon A \times \{f_0 b\}$]

(4) <u>If</u> τ'' says "$A \subseteq \mathcal{O}^F$ holds"

<u>Then</u> we are done

Now assume that τ'' does not say "$A \subseteq \mathcal{O}^F$"

Then for some $a_i \varepsilon A$ we have

$[a_i \notin \mathcal{O}^F] \vee [a_i \varepsilon \mathcal{O}^F \; \& \; |a_i| > \tau'']$.

As $\tau'' > \min\{|a_i|, |b_j|\}$ where a_i is the "bad one" we just described, and b_j is <u>any</u> member of B, it must be the case that

τ'' tells us that every member of B is in $\mathcal{O}^F$

$\therefore$ $b \varepsilon \mathcal{O}^F$ and $\tau'' \geq |b|$

Assume $<q,a*b>$ is the notation we got for τ''.

Let then $f(a,b) = q$.

Remark 0.6

It is obvious that we can generalize this result and prove by induction on the number of (candidate) notations taken in consideration that for every integer k, there is a partial recursive function $f : I^k \to \omega$ such that if at least one of $a_1,\ldots,a_k \in \mathcal{O}^F$, then $f(a_1,\ldots,a_k)$ is defined, say $f(a_1,\ldots,a_k) = i \in \omega$ and

(1) $<i,a_1*\ldots*a_k> \in \mathcal{O}^F$ and

(2) $|<i,a_1*\ldots*a_k>| \geq \min\{|a_1|,\ldots,|a_k|\}$.

Nevertheless we would like to note here that the proof of theorem 0.5 can be rewritten in order to get a proof of this remark, for every k, not using the induction mentioned a few lines earlier.

In this new proof, the only interesting modification comes from case 4: in theorem 0.5 we used "at most 3 ordinals" to compare 2 notations, we shall now need at most k+1 ordinals because of the following claim:

Claim (3)

If $\ell \leq k$, and we have already obtained (effectively) an ordinal τ (i.e.: we have an index for τ) such that τ shows that $f_0a_1,\ldots,f_0a_\ell \varepsilon \mathcal{O}^F$, but does not say anything about the sets $A_1,\ldots,A_\ell$ defined respectively by $a_1,\ldots,a_\ell$.

Then we can (effectively) get an ordinal τ' (i.e. an index t for τ') such that $\tau' > \tau$ and either τ' tells us that one of the sets A_i is a subset of $\mathcal{O}^F$ $(1 \leq i \leq \ell)$ or τ' tells us that one (at least) of $f_0a_{\ell+1},\ldots,f_0a_k \varepsilon \mathcal{O}^F$.

Proof of the claim

1) We get τ' (and t) as in theorem 0.5 by comparing all possible k-tuple in $A_1\times\ldots\times A_\ell\times\{f_0a_{\ell+1}\}\times\ldots\times\{f_0a_k\}$

2) Assume that τ' does not establish that $A_i \subseteq \mathcal{O}^F$ for any $1 \leq i \leq \ell$, then for some choice of $x_1,\ldots,x_\ell$ in $A_1,\ldots,A_\ell$ respectively we shall have

(1) $f(x_1,\ldots,x_\ell,f_0a_{\ell+1},\ldots,f_0a_k)$ is defined

(say $f(x_1,\ldots,x_\ell,f_0a_{\ell+1},\ldots,f_0a_k) = i$)

(2) $<i,x_1*\ldots*x_\ell*f_0a_{\ell+1}*\ldots*f_0a_k> \varepsilon \mathcal{O}^F$

(3) $|<i,x_1*\ldots*x_\ell*f_0a_{\ell+1}*\ldots*f_0a_k>| = \alpha < \tau'$

(4) $\alpha \geq \min\{|x_1|,\ldots|x_\ell|,|f_0a_{\ell+1}|,\ldots,|f_0a_k|\}$

$\therefore$ τ' must recognize one of $x_1,\ldots,x_\ell,f_0a_{\ell+1},\ldots,f_0a_k$ as a member of $\mathcal{O}^F$; but τ' is unable to say whether $x_1,\ldots,x_\ell \in \mathcal{O}^F$ or not. Hence it must be the case that τ' is able to tell us that one of $f_0a_{\ell+1},\ldots,f_0a_k \in \mathcal{O}^F$

claim (3) ⊣

The proof of case 4 of remark 0.6 is then (nearly) identical to that of case 4 of theorem 0.5 after replacing claim 1 and claim 2 by claim 3 (for $\ell=0$ and $\ell\neq 0$ respectively). Thus when we start, we get an ordinal τ' and τ' must tell us that something (say f_0a_1) is a notation. Then τ^2 will either say that $A_1 \subseteq \mathcal{O}^F$, and then we are done, or only establish that another candidate (say f_0a_2) is a member of $\mathcal{O}^F$.

Assume the worst happens: we must go up to τ^k. Then we know that all the f_0a_i's $(1 \leq i \leq k)$ are notations but we still do not know anything about the sets A_i's $(1 \leq i \leq k)$. Then we can get τ^{k+1} and, by claim 3, τ^{k+1} must give us some new information: hence the only possibility for τ^{k+1} is to establish that one of the A_i's $(1 \leq i \leq k)$ is a subset of $\mathcal{O}^F$. ⊣

We are now ready to prove the next theorem, first announced by Grilliot in [7] and whose first correct proof is due to Harrington and MacQueen (see [15])

Theorem 0.7 (Grilliot)

If (1) $J \subseteq I$ is recursive in F

(2) ${}^J I = \{f \mid f$ is a function from J to $I\}$ is a subset of I

(3) ${}^J I$ is recursive in F

Then there exists a partial recursive function $f: {}^J I \to \omega$ such that, for every $a \in {}^J I$ if $a(j) \in \mathcal{O}^F$ for some $j \in J$, then fa is defined, $\langle fa,a \rangle \in \mathcal{O}^F$ and $|\langle fa,a \rangle| \geq \min \{|a(j)| \mid j \in J\}$.

The proof uses (heavily) the recursion theorem, an effective transfinite induction and the proof of Gandy's theorem given in remark 0.6.

Remark 1

This theorem needs a little bit of choice. From now on we will always assume that AC holds. (AC has not been used in the proof of 0.5 or 0.6).

Remark 2

We assume that J is recursive in F and ${}^J I \subseteq I$

$\therefore$ ${}^{J}I$ is a subset of I recursive in F (i.e. (3) in the hypothesis follows from (1) and (2)) and the power set of J is recursive in F.

Look at all the subsets of J which are prewellorderings (no choice needed here); partition this subset of 2^{J} using the equivalence relation "... is of same length as...". This gives a <u>recursive</u> prewellordering of I [namely the sup of all ...] of length $(\text{card } J)^{+}$

<u>Proof</u>

The proof is by induction on the min:

<u>induction hypothesis</u> $(a, b \in {}^{J}I)$

| If $\min\{|b(j)| \mid j \in J\} < \min\{|a(j)| \mid j \in J\}$
| <u>Then</u> fb is defined, $\langle fb,b\rangle \in \mathcal{O}^{F}$ and
| $|\langle fb,b\rangle| \geq \min\{|b(j)| \mid j \in J\}$

<u>Remark 3</u>

In theorem 0.5 we tried to compare two (candidate) notations for ordinals, in remark 0.6 we extended the proof to k notations, for any $k \in \omega$. Here we shall extend the result to "J-many" notations.

<u>Case 1</u> One of the $a(j)$'s is of the form $\langle 1,a'\rangle$

Then let $f(a) = 1$.

Case 2 All the $a(j)$'s are of the form $\langle 2^{n_j}, a'(j)\rangle$

We assume that at least one of the $a(j)$'s is in $\mathcal{O}^F$, hence we have: at least one of the $f_0(a(j))$'s is in $\mathcal{O}^F$ and

$$\min \{|f_0(a(j))| \mid j \in J\} < \min \{|a(j)| \mid j \in J\}$$

Let $b \in {}^J I$ be defined as follows:
$b(j) = f_0(a(j))$. Thus by induction hypothesis, fb is defined and $\langle fb,b\rangle \in \mathcal{O}^F$. Say $|\langle fb,b\rangle| = \tau$

$\therefore$ H_τ will recognize at least one of the $f_0(a(j))$'s as an element of $\mathcal{O}^F$

$\therefore$ We can get effectively a notation from a for $\tau+1$, say it is $\langle p,a\rangle$. Let $f(a) = p$.

[Note that p must be an integer of the form 2^q where q carries all the information contained in $f(b)$, together with the instructions needed to go from a to b].

Case 3 For some $j \in J$ (perhaps many), $a(j)$ looks like a notation for a successor ordinal and for some $j_1 \in J$ (perhaps many), $a(j_1)$ looks like a notation for a limit ordinal. Assume that for some j, $a(j) \in \mathcal{O}^F$.

Then, take τ as before, by looking at the derived notations.

Claim (4)

τ must recognize at least one of the derived notations as an element of $\mathcal{O}^F$.

Proof of the claim

By induction hypothesis, $\tau \geq \min\{|f_0(a(j))| \mid j \in J\}$ and by hypothesis $(\exists j \in J)(f_0(a(j)) \in \mathcal{O}^F)$. Apply fact G.

claim (4) |

If one of the derived notations recognized by τ is derived from a candidate notation for a successor, then we are done (by case 2). So let's assume that we deal only with candidate notations for a limit ordinal, which is case 4.

Case 4 Assume that all the $a(j)$'s are of the form $<3^{n_j}.5^{e_j},a'_j>$ and that, for some $j \in J$, $a(j) \in \mathcal{O}^F$.

(5) Let $A_j = \{x \in I \mid <I,H|_{<n_j,a'_j>}|> \models \phi_{e_j}(x,a'_j)\}$

[Note that this definition (5) has a meaning iff $<n_j,a'_j> = f_0(a(j)) \in \mathcal{O}^F$].

Stage 0 by induction hypothesis we can get a notation for an ordinal $\tau^0 \geq \min\{|f_0(a(j))| \mid j \in J\}$. Hence τ^0 must recognize at least one $f_0(a(j))$ as a member of $\mathcal{O}^F$. For each $f_0(a(j))$ recognized by τ^0, develop the set A_j. (It is possible, because τ^0 is big enough to use (5): this works as in the proof of remark 0.6)

Stage $\alpha+1$

Assume that up to stage α, we have got hold (effectively) of ordinals $\tau^0, \tau^1, \ldots, \tau^\alpha$ (with $\tau^0 < \tau^1 < \ldots < \tau^\alpha$) which have, each in his turn, recognized some $f_0(a(j))$ as a member of $\mathcal{O}^F$. (Say τ^α does it for all $j \in J_0$). Assume also that we are still unable to say anything about the $f_0(a(j))$'s with $j \in J - J_0$, but that we have developed all the sets A_j for $j \in J_0$, not knowing whether $A_j \subseteq \mathcal{O}^F$ or not.

We know that one of the $a(j)$'s, $j \in J$, is a notation (by hypothesis).

Define $\tau^{\alpha+1}$ as follows: pick in all possible ways one element in each set A_j $(j \in J_0)$ and complete the sequence by the $f_0(a(j))$ for $j \in J - J_0$. By induction hypothesis, for each "picking" p, we get (effectively) an ordinal $\tau(p)$. Take (effectively) a notation for the first limit ordinal strictly bigger than all $\tau(p)$ and τ^α: it is a notation for $\tau^{\alpha+1}$.

(1) This procedure is effective (i.e. $\tau^{\alpha+1}$ has a notation) because we took all possible "pickings".

(2) As we want to be able to "pick", we need choice. This does not mean that we will be able to get a recursive well ordering of I.

Claim (6)

Either $\tau^{\alpha+1}$ identifies (at least) one of the A_j's $(j \in J_0)$ as a subset of $\mathcal{O}^F$ (which τ^α could not do) or $\tau^{\alpha+1}$ recognizes (at least) one $f_0(a(j))$, $j \in J - J_0$, as a member of $\mathcal{O}^F$ (which could not be done by τ^α).

Proof of the claim (as in remark 0.6)

Assume $\tau^{\alpha+1}$ does not identify any A_j $(j \in J_0)$ as a subset of $\mathcal{O}^F$. Hence there is a "picking" of members of the A_j's $(j \in J_0)$ such that for each element b' chosen in, say A_j, we have $[b' \notin \mathcal{O}^F] \vee [b \in \mathcal{O}^F \ \& \ |b'| > \tau^{\alpha+1}]$. Hence, by induction hypothesis, $\tau^{\alpha+1}$ must be bigger or equal to some $|f_0(a(j))|$ for some $j \in J - J_0$ and thus, $\tau^{\alpha+1}$ recognizes this $f_0(a(j))$ as a member of $\mathcal{O}^F$.

claim (6) |

<u>Stage λ</u> , λ limit , $\lambda < (\text{card } J)^+$

Here appears the main difference with remark 0.6: let τ^λ be the first limit ordinal strictly bigger than the effective supremum of $\{\tau^\beta \mid \beta \in \lambda\}$. Remember that for each such τ^β we have a notation. Here we need the fact that ${}^J I$ is recursive in F, thus that there is a <u>recursive</u> prewellordering of I of length $(\text{card } J)^+$ to justify the fact that we can go <u>effectively</u> as far as τ^λ.

As $\tau^{\alpha+1}$ is always doing something new ($\alpha < (\text{card } J)^+$) by claim (6), we will eventually (as in remark 0.6) establish that some A_j is a subset of $\mathcal{O}^F$, or (in at most (card J)-steps) that all the $f_0(a(j))$ are in $\mathcal{O}^F$: at the next step we must then show that some A_j ($j \in J$) is a subset of $\mathcal{O}^F$.

Hence we proved that a(j) is a notation and we have an ordinal τ, and a notation from a, <q,a>, for τ (all obtained effectively) such that $\tau = |\langle q,a\rangle| \geq |a(j)|$.

Let then $f(a) = q$.

Note that Tp(n-1) has all the properties required for J in theorem 0.7.

Grilliot's theorem has as immediate consequence the following results:

Corollary 0.8

The relations semirecursive in F are closed under existential quantification over Tp(n-1)

Corollary 0.9

Let R be a non empty set of subindividuals semirecursive in F. Then there exists a non empty set $S \subseteq R$ recursive in F. We can even compute S uniformly from an index (h-index) for R.

The following theorem summarizes the most interesting properties of the sets semirecursive in F.

Theorem 0.10 (Assume $n > 0$)

Sets semirecursive in F are

a) closed under universal quantification over I

b) not closed under existential quantification over I

c) closed under existential quantification over Tp(n-1) (hence over ω)

Definition 0.6

[Remember that we are still in Harrington's universe. Similar definition (see definition 0.5) could be given for Kleene's or Sacks' frame].

The following ordinals are naturally associated with F:

(i) $\kappa_0^F = \sup \{|\langle m,0\rangle| \mid \langle m,0\rangle \in \mathcal{O}^F\}$

= sup of the prewellorderings of I which are recursive in F

= sup of the ordinals having an integer as notation.

(ii) For $j \leq n$, we have more generally:

$\kappa_j^F = \sup \{|\langle m,a\rangle| \mid a \in Tp(j) \ \& \ \langle m,a\rangle \in \mathcal{O}^F\}$

$= \sup \{\kappa_0^{\langle F,a\rangle} \mid a \in Tp(j)\}$

It is easy to see that, for $n > 0$ and for all $j < n$, there are ordinals $\sigma < \kappa_j^F$ which do not have a notation from a type j-object (although they have notations). This motivates the definition of another ordinal:

(iii) λ_j^F = sup of all the prewellorderings of $Tp(j)$ which are recursive in $\langle F,a\rangle$ for some $a \in Tp(j)$

= the order type of $\{|\langle m,a\rangle| \mid a \in Tp(j) \ \& \ \langle m,a\rangle \in \mathcal{O}^F\}$

We will now present a set theoretic frame for a definition of recursion in F.

<u>Definition 0.7</u> (Harrington)

The L(F)-hierarchy

For an ordinal σ, define $L_\sigma(F)$ by:

$L_0(F) = I$

$L_{\sigma+1}(F) = \{X \subseteq L_\sigma(F) \mid X$ is first order definable with parameters over the structure $M_\sigma(F)\}$

$L_\lambda(F) = {}_{\delta<\lambda}\, L_\delta(F)$ for λ limit

with $M_\sigma(F) = \langle L_\sigma(F), \varepsilon, F \restriction (Tp(n+1) \cap L_\sigma(F))\rangle$

Let $\mathcal{L}$ be the first order language appropriate to the structures $M_\sigma(F)$.

---

1. A few equivalences

The L(F)- hierarchy is in fact very similar to the hierarchy used to define "h-recursive in F". We shall show later that these two frames are (more or less) equivalent.

In fact we dispose now of 4 frames: the L(F)-hierarchy and the three "universes" of, respectively and chronogically: Kleene, Sacks and Harrington ([10], [24], [8]). The first and the last one seem to be equivalent: we want to state precisely, and prove, what this means. We will also show that, if we assume that $F = \langle F', {}^{n+2}E \rangle$ where F' is some fixed object of type n+2, the definitions of "h-recursive in F", "k-recursive in F" and "s-recursive in F" are equivalent (We assume $n \geq 1$).

We would like first to use the setting of the L(F)-hierarchy to define a (fourth) notion of "recursive in F". The main problem is due to the fact that the three other frames yield to a collection of "interesting ordinals", we have even for each of these universes a top-ordinal, the biggest interesting ordinal, which we ambiguously denoted always by κ^F; but the L(F)-hierarchy does not give us the least indication about the height of this κ^F.

Assume (provisionally) that we know the exact value of κ^F and of the ordinals $\kappa_0^{<F,G>}$ for all $G \in Tp(n+2)$. We might want to give now the following definitions, in the frame of the L(F)-hierarchy.

<u>Definition 1.1</u>

For all $G \in Tp(n+2)$, $\{e\}^{<F,G>}(0) \downarrow$ (converges) iff $M_{\kappa_0^{<F,G>}}(<F,G>) \models \phi_e$ where ϕ_e is the $e^{\underline{th}}$ Σ_1 sentence over $M_{\kappa_0^{<F,G>}}(<F,G>)$.

For $a \in I$, $M_\sigma(<F,a>)$ is essentially the same as $M_\sigma(F)$. Thus <u>fixing F</u>, we define now:

<u>Definition 1.2</u>

$A \subseteq I$ is semirecursive in F iff there is a Σ_1 formula $\phi(x)$ in L such that for all $a \in I$

$$[a \in A \text{ iff } M_{\kappa_0^a}(F) \models \phi(a)]$$

Luckily, if we want to define in a natural way a recursion theory in the L(F)-frame, we do not need to give all the κ^F and $\kappa_0^{<F,G>}$: it is enough to translate the definition of h-recursion. We will show first that for $\sigma < \kappa^F$:

1) $M_\sigma(F)$ can be coded by a subset of I primitive recursive in $H^F_{\sigma+2}$ and

2) H^F_σ is first order definable over $M_\sigma(F)$. (and this can be done uniformly).

We will then prove the equivalence between the other definitions as follows:

3) s-recursive is equivalent to h-recursive.

4) h-recursive implies k-recursive

5) k-recursive implies s-recursive.

Proposition 1.1

For each notation $a \in \mathcal{O}^F$, say $|a|^F = \sigma$, we define uniformly in a 5 subsets of I: X_a, E_a, F_a, $=_a$, P_a and one subset of $\omega \times I$: S_a all primitive recursive in $\langle H^F_{\sigma+2}, a\rangle$ such that the structure $M_\sigma(F)$ will be coded by these sets and relation where X_a will code the universe $L_\sigma(F)$, with P_a as equivalence relation (two different individuals could denote the same member of $L_\sigma(F)$), E_a will code the ε relation and F_a the relation "Fx = y". Finally $=_a$ will code the equality and S_a the satisfaction predicate.

In fact, for each $a \in \mathcal{O}^F$ we will define a code M_a for a model of $M_{|a|^F}(F)$. We must thus keep track inside each of these M_a's of the different "names" given to the same object and of the parallel development of all the M_a's coding the single $M_{|a|^F}(F)$.

We will define a coding in such a way that if a set $\tilde{x}$ is constructed in the $L(F)$- hierarchy at level $\sigma < \kappa^F$ (say $\sigma = |a|^F$), then $\tilde{x}$ will be coded by a pair of the form <a,m>.

As for each $a \in \mathcal{O}^F$ we will define a model M_a to encode $M_{|a|^F}(F)$, if $a, b \in \mathcal{O}^F$ & $|a|^F = |b|^F = \sigma$, then for each member of $L_\sigma(F)$, M_a and M_b will have different codes. This explains why we need an equivalence relation to keep track of these different codes for a given element of $L(F)$. As we wish to take unions over ordinals (hence over all possible notations for such ordinals) at the limit stages, we will have inside a given model M_a (say $|a|^F = \omega+\omega$) many different codes for the same object of $L(F)$. It is thus clearly a necessity to keep track of the relation "x and y are codes from a for the same object of $L_{|a|^F}(F)$".

As earlier, we will use the * notation to make some formulas more readable: <a,b*c> must be read as <a,<b,c>> where < > is our "universal" coding operator.

Notations.

1) If x a code ($x \varepsilon X_a$ for some $a \varepsilon \mathcal{O}^F$)

Then $\tilde{x}$ is the element of $L_{|a|}(F)$ coded by x

and x is said to be a code from a

2) If $a \varepsilon \mathcal{O}^F$, then $a' = \langle 2^{(a)_0}, (a)_1 \rangle$ so that

$a' \varepsilon \mathcal{O}^F$ and $|a'|^F = |a|^F + 1$

3) If $b \varepsilon \mathcal{O}^F$ and $|b|^F = \tau+1$

Then $b-1$ is an abreviation for $\langle \log_2 (b)_0, (b)_1 \rangle$

i.e. $b-1$ represents a notation for τ from $(b)_1$.

Proof of proposition 1.1

The proof is by induction on $|a|^F$

Case 0 $\quad |a|^F = 0$

$\therefore$ a is of the form $\langle 1,b \rangle$

$M_0(F) = \langle I, \varepsilon, F \restriction I \rangle$

Let (i) $x \varepsilon X_a \iff (x)_0 = a \;\&\; (x)_1 \varepsilon I$

[here $\tilde{x} = (x)_1$]

(ii) $(x,y) \varepsilon E_a \iff x \varepsilon X_a \;\&\; y \varepsilon X_a \;\&\; (x)_1 \varepsilon (y)_1$

(iii) $(x,y) \varepsilon F_a \iff x \varepsilon X_a \;\&\; y \varepsilon X_a \;\&\; (y)_1 \varepsilon \omega$

$\&\; (\exists c)(\exists e)[e \varepsilon \omega$ & c is a finite sequence of individuals

$\&\; (x)_1 = \{e\}_p^{\langle H_0^F, c \rangle} \;\&\; \langle e, c, (y)_1 + 1 \rangle \varepsilon H_1^F]$

(iv) $(x,y) \varepsilon P_a \iff (\exists b)(\exists c)[(x)_0 = \langle 1,b\rangle$ & $(y)_0 = \langle 1,c\rangle$ & $(x)_1 = (y)_1]$

(v) $(x,y) \varepsilon =_a \iff x \varepsilon X_a$ & $y \varepsilon Y_a$ & $(x)_1 = (y)_1$

(vi) Satisfaction

Let e be the Gödel number of ϕ.

Let b be a finite sequence of members of X_a.

By (i), (ii), (iii), (iv) and (v) we know that X_a, E_a, F_a, P_a and $=_a$ are first order definable over $\langle I, H_1^F\rangle$.

Let ϕ_0, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 be the formulas which, respectively, define them. Then

$\langle X_a, E_a, F_a, =_a, P_a\rangle \models \phi_e(b)$ iff $\langle I, H_1^F\rangle \models \phi_{e^*}((b)_1)$,

where e^* carries all the information contained in e and the instructions needed to translate ϕ_e using ϕ_0, ϕ_1, ϕ_2, ϕ_3 and ϕ_4. But $\langle I, H_1^F\rangle \models \phi_{e^*}((b)_1)$ iff $(b)_1 \varepsilon W_{e^*}^{H_1^F}$ iff $\langle e^*, (b)_1, 0\rangle \varepsilon H_2^F$.

Remarks (Remember that $|a|^F = 0$)

• (i) means that x will be a code from a for an individual b iff $x = \langle a,b\rangle$

• (ii) means that to know whether a pair of individuals is in E_a (which attempts to code ε) we simply look at the objects which are coded: since we are at level 0, this can be done easily.

• (iii) means that if x is a code from a for $\tilde{x}$ and if y is a code from a for $\tilde{y}$, then $F\tilde{x} = \tilde{y}$ holds iff $\tilde{y} \varepsilon \omega$ and $\tilde{x}$, viewed as a primitive recursive function, argument of F, yields to $\tilde{y}$. Here again we use the fact that $\tilde{x} = (x)_1$.

• (iv) means that if x is a code from $<1,b>$ and y is a code from $<1,c>$ (with $b, c \varepsilon I$), then "$|a|^F$ knows whether they code the same element of $L_{|a|^F}(F)$, or not": again we just need to look at $(x)_1$ and $(y)_1$.

• (v) means that if x and y are both codes from a, then $|a|^F$ knows whether $\tilde{x} = \tilde{y}$ or not: again we simply look at $(x)_1$ and $(y)_1$.

<u>Case 1</u> $a \varepsilon \mathcal{O}^F$ and $|a|^F = \sigma+1$

a is of the form $<2^e,u>$

$b = <e,u>$ is an index for σ, $a' = <2^{2^e},u>$ is an index for $\sigma+2$ and $a'' = <2^{2^{2^e}},u>$ is an index for $\sigma+3$.

But $(a)_1 = (b)_1 = (a')_1 = (a'')_1 = u$.

Let

(i) $(x,y) \varepsilon E_a \Leftrightarrow [(x,y) \varepsilon E_b] \vee [(x,y) \notin E_b \;\&$

$x \varepsilon X_b \;\&\; y \notin X_b \;\&\; (\exists k \varepsilon \omega)(\exists c)(y = \langle a, k^*c\rangle \;\&$

c is a finite sequence of members of X_b &

k is the least Gödel number of some formula ϕ &

k defines a new set & $S_b(k, x^*b)$]

i.e.: $(x,y) \varepsilon E_a \Leftrightarrow (x,y)$ was already in E_b, the "preceding" code for the ε-relation, or this was not the case, but x was in X_b, $y = \{u \varepsilon X_b \mid \phi_k(u,c)$ holds in $M_b\}$ for some formula ϕ_k and c a finite sequence of elements of M_b) and $M_b \models \phi_k(x,c)$.

(ii) $y \varepsilon X_a \Leftrightarrow [y \varepsilon X_b] \vee [y \notin X_b \;\&\; (\exists x)$

$(x \varepsilon X_b \;\&\; (x,y) \varepsilon E_a)]$

i.e.: X_a is the field of E_a

(iii) $(x,y) \varepsilon F_a \Leftrightarrow [(x,y) \varepsilon F_b] \vee [(x,y) \notin F_b$

$\&\; x \varepsilon X_a \;\&\; y \varepsilon X_a \;\&\; (y)_1 \varepsilon \omega \;\&\; (\exists k \varepsilon \omega)(\exists c \varepsilon X_a)$

$((x)_1 = \langle k,c\rangle \;\&\; \langle k^*, c, (y)_1 + 1\rangle \varepsilon H^F_{\sigma+2})]$

(i.e.: $(x,y) \varepsilon F_a$ iff (x,y) was already in F_b or this was not the case but x is in X_a, defined using the formula ϕ_k and the parameter(s) c, y is also in X_a and codes an element of ω and $F(\tilde{x}) = \tilde{y}$. To express this last relation, we would like to use immediately $H^F_{\sigma+1}$, but $<i,d,m+1> \varepsilon H^F_{\sigma+1}$ iff $<I,H^F_\sigma> \models F(\{i\}_p^{<H^F_\sigma,d>}) = m$, so we must first use the definition that we have of x, y and c to translate the formula. This explains why we use k^* instead of k. We must then use $H^F_{\sigma+2}$ because we started with $|a|^F = \sigma+1$, thus: we have "$<k^*,c,(y)_1+1> \varepsilon H^F_{\sigma+2}$"

(iv) $(x,y) \varepsilon P_a \Leftrightarrow [(x,y) \varepsilon P_b] \vee [(x,y) \notin P_b \ \& \ (x)_0 \notin O^F_{\sigma+1} \ \& \ (x)_0 \varepsilon O^F_{\sigma+2} \ \& \ (y)_0 \notin O^F_{\sigma+1} \ \& \ (y)_0 \varepsilon O^F_{\sigma+2} \ \& \ (\exists k,k' \varepsilon \omega)$ $(\exists c,c' \varepsilon I) \ (x = <(x)_0,k*c> \ \& \ y = <(y)_0,k*c'> \ \&$ $(\forall u \varepsilon X_{(x)_0-1}) (\exists v \varepsilon X_{(y)_0-1}) (S_{(x)_0-1}(k,u*c) \Rightarrow$ $S_{(y)_0-1}(k^*,<v*c',u>)) \ \& \ (\forall v \varepsilon X_{(y)_0-1})$ $(\exists u \varepsilon X_{(x)_0-1}) (S_{(y)_0-1}(k',v*c') \Rightarrow S_{(x)_0-1}(k^{**},<u*c,v>)))]$

where k^* is a Gödel number for "$(u,v) \varepsilon P_b \ \& \ \phi_{k'}(v,c')$" and k^{**} is a Gödel number for "$(v,u) \varepsilon P_b \ \& \ \phi_k(u,c)$" i.e. $(x,y) \varepsilon P_a$ (or in other terms: $|a|^F$ sees that $\tilde{x}$ and $\tilde{y}$ are equal, even if they are not codes from the same individual) iff they are already considered as such before $|a|^F$, or this was not the case (because they are both new) and $(x)_0$, $(y)_0$ are both notations for $\sigma+1$, and for all u satisfying the definition of x, there is a v satisfying the definition of y (in their respective models) and $|b|^F$ says that this v is equivalent to the given u ("equivalent" for $|b|^F$ is already defined); and conversely...

$\therefore$ every element of $X_{(x)_0}$ satisfying the definition of x in $M_{(x)_0}$ has a "perfect" copy (at least one, probably many) in $X_{(y)_0}$, satisfying the definition of y in $M_{(y)_0}$ and conversely: every...

• (v) $(x,y) \in =_a \iff [x \in X_b \ \& \ y \in X_b \ \& \ (x,y) \in =_b]$

$\vee\ [x \notin X_b \ \& \ x \in X_a \ \& \ y \notin X_b \ \& \ y \in X_a \ \& \ (\exists k_1, k_2 \in \omega)$

$(\exists c_1, c_2 \in I)\ (x = \langle a, k_1 * c_1\rangle \ \& \ y = \langle a, k_2 * c_2\rangle$

$\& \ S_b(\langle k_1 {}^* k_2\rangle, c_1 * c_2))]$

Where $\langle k_1 {}^* k_2\rangle$ is a Gödel number for the formula "$(\forall u)(\phi_{k_1}(u,c_1) \iff \phi_{k_2}(u,c_2))$".

i.e. $(x,y) \in =_a$ iff <u>both x and y are in X_a</u> and either they are already in X_b and "$(x,y) \in =_b$" holds or none of them is in X_b, in which case x is defined via some Gödel number k_1 and the parameter(s) c_1 and y is defined via some Gödel number k_2 and parameter(s) c_2. k_1 and k_2 have been chosen to be the smallest Gödel number for the formula defining x from c_1 (respectively y from c_2), but this does not imply that ϕ_{k_1} and ϕ_{k_2} are the same (e.g.: let ϕ_{k_1} be $(u = 2+2)$ and ϕ_{k_2} be $(u = 2^2)$, ϕ_{k_1} and ϕ_{k_2} define the same set). The only thing we must say is that the same elements of X_b satisfy $\phi_{k_1}(u,c_1)$ and $\phi_{k_2}(u,c_2)$ in M_b; this can be done using a Gödel number obtained effectively from k_1 and k_2, and the satisfaction predicate S_b (which is primitive recursive in $\sigma+2 = |a|^F+1$).

(vi) Satisfaction

We know that $|a|^F = \sigma+1$, $|b|^F = \sigma$ and $(a)_1 = (b)_1$

Let e be the Gödel number of ϕ and let d be a finite sequence of members of X_a.

We know already that X_a, E_a, F_a, P_a, $=_a$ are first order definable over $<I,H^F_{\sigma+2}>$. Let ϕ_0, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 define respectively these sets over $<I,H^F_{\sigma+2}>$. Let ϕ_5 define d.

Then $\langle X_a, E_a, \ldots\rangle \models \phi_e(d)$ (i.e. $S_a(e,d)$)

iff $\langle \mathcal{I}, H^F_{\sigma+2}\rangle \models \phi_{e^*}(d)$

where e^* carries all the information contained in e and the instructions needed to translate ϕ_e using $\phi_0, \phi_1, \phi_2, \phi_3, \phi_4$ and ϕ_5.

But $\langle \mathcal{I}, H^F_{\sigma+2}\rangle \models \phi_{e^*}(d) \iff d \in W^{H^F_{\sigma+2}}_{e^*}$

$\iff \langle e^*, d, 0\rangle \in H^F_{\sigma+3}$

<u>Case 2</u> $|a|^F = \lambda$, λ limit ordinal

$\therefore$ a is of the form $\langle 3^e \cdot 5^m, a'\rangle$

At this stage we will attempt to glue together all the M_b's for $b \in \mathcal{O}^F$, such that $|b|^F < \lambda$.

(i) $x \in X_a \iff (\exists b \in \mathcal{O}^F_\lambda)(x \in X_b)$

i.e.: $X_a = \bigcup\{X_b \mid |b|^F < \lambda\}$

(ii) $(x,y) \in P_a \iff (\exists b \in \mathcal{O}^F_\lambda)[(x,y) \in P_b]$

i.e. λ says that x and y are code for the same object iff it has already been said before.

(iii) $(x,y) \in E_a \iff (\exists b \in \mathcal{O}^F_\lambda)(\exists u,v)(u \in X_b \ \& \ v \in X_b \ \&$

$(u,v) \in E_b \ \& \ (x,u) \in P_b \ \& \ (y,v) \in P_b)$

i.e.: $(x,y) \varepsilon E_a$ iff somewhere before some $|b|^F$ said that x is equivalent to some u, y is equivalent to some v, u and v belong both to X_b and $\tilde{u} \varepsilon \tilde{v}$ is true in $M_{|b|}(F)$

(iv) $(x,y) \varepsilon F_a$ iff $(\exists b \varepsilon \mathcal{O}_\lambda^F)(\exists u,v)(u \varepsilon X_b \ \& \ v \varepsilon X_b \ \&$
$(u,v) \varepsilon F_b \ \& \ (x,u) \varepsilon P_b \ \& \ (y,v) \varepsilon P_b)$

(v) $(x,y) \varepsilon =_a$ iff $(x,y) \varepsilon P_a$

(vi) Satisfaction

Let e be a Gödel number of ϕ and d a finite sequence of members of X_a.

We know already that X_a, E_a, ... are first order definable over $\langle I, H_{\lambda+1}^F \rangle$. Let $\phi_0, \ldots, \phi_4$ define respectively these sets over $\langle I, H_{\lambda+1}^F \rangle$. Let ϕ_5 define d.

Then $S_a(e,d) \Longleftrightarrow \langle X_a, E_a, \ldots \rangle \models \phi_e(d)$

$\Longleftrightarrow \langle I, H_{\lambda+1}^F \rangle \models \phi_{e^*}(d) \Longleftrightarrow \langle e^*, d, 0 \rangle \varepsilon H_{\lambda+2}^F$

where e^* is obtained as in the successor case.

[In fact as λ is limit, and as we only want to quantify over all $b \varepsilon \mathcal{O}^F$ such that $|b|^F < \lambda$, we could start with sets first order definable over $\langle I, H_\lambda^F \rangle$ and end with $\langle e^*, d, 0 \rangle \varepsilon H_{\lambda+1}^F$. We chose to start with definitions over $\langle I, H_{\lambda+1}^F \rangle$, to have for all $\sigma < \kappa^F$: $M_\sigma(F)$ can be coded by a set primitive recursive in $H^F_{\sigma+2}$ and a notation for σ.]

Thus at the limit stage, either we want to look at some previous stage, and to do this we may want to replace everything we have by equivalent copies, all in the same model (where the equivalence is vouched for by some P_b) and we do not add anything new to the set considered, or (case of $=_a$) we take all equivalent objects using all P_b's (for $|b|^F < |a|^F$) in this definition. As we introduce at the limit level many equivalent objects in the same model (previously they were in completely different ones) we must enlarge considerably the relation coding the equality.

The satisvaction is defined in the obvious way. ⊣

Remark

The proof of proposition 1.1 uses constantly lemma 0.2, corollary 0.3 and our fact G!!!

Proposition 1.2

For each $\sigma < \kappa^F$, given a notation for σ, H^F_σ and $\mathcal{O}^F_\sigma$ are first order definable over $M_\sigma(F)$.

In fact we shall prove (by induction) that there are functions f and g such that:

<u>If</u> $\langle e,a\rangle$ is an h-index for σ ,

<u>Then</u> $f(\langle e,a\rangle)$ is defined and is the Gödel number of a formula $\phi_{f(\langle e,a\rangle)}$ of L such that $\phi_{f(\langle e,a\rangle)}$ defines H^F_σ over $M_\sigma(F)$ and $g(\langle e,a\rangle)$ is defined and is the Gödel number of a formula $\phi_{g(\langle e,a\rangle)}$ of L such that $\phi_{g(\langle e,a\rangle)}$ defines $\mathcal{O}^F_\sigma$ over $M_\sigma(F)$ (where $\mathcal{O}^F_\sigma$ is as usual $\{b \in \mathcal{O}^F \mid |b|^F < \sigma\}$)

<u>Case 0</u> Let $\langle 1,a\rangle$ be an h-index for 0

Let e_0 be a Gödel number for $x \neq x$

Then, let $f(\langle 1,a\rangle) = g(\langle 1,a\rangle) = e_0$

<u>Case 1</u> Let $\langle 2^e,a\rangle$ be an h-index for $\sigma+1$

$\therefore$ $\langle e,a\rangle$ is an h-index for σ

Then $H^F_{\sigma+1} = \{\langle i,b,0\rangle \mid b \in W_i^{H^F_\sigma}\} \cup \{\langle i,b,x+1\rangle \mid F(\{i\}_p^{\langle H^F_\sigma,b\rangle}) = x\}$

$b' \in H^F_{\sigma+1} \iff (\exists i,y \in \omega)(\exists b \in I)([(y = 0\ \&$

$\{i\}_p^{H^F_\sigma}(b) = 0) \vee (\exists x \in \omega)(y = x+1\ \&\ F(\{i\}_p^{\langle H^F_\sigma,b\rangle}) = x)]$

$\&\ b' = \langle i,b,y\rangle)$

But 1) $H^F_\sigma \subseteq L_\sigma(F)$ and we have an index for σ

H^F_σ is first order definable over $M_\sigma(F)$ by induction hypothesis

$\therefore$ $\phi_{f(<e,a>)}$ defines H_σ^F over $M_\sigma(F)$

$\therefore$ $H_\sigma^F \varepsilon L_{\sigma+1}(F)$

and 2) By definition $\{e\}_p^X$ is the $e^{\underline{th}}$ function which is first order definable over $<I,X>$

$\therefore$ $\{i\}_p^{H_\sigma^F}$ is first order definable over $M_\sigma(F)$

$\therefore$ $\{i\}_p^{H_\sigma^F} \varepsilon L_{\sigma+1}(F)$

Hence we can express the relation "$b' \varepsilon H_{\sigma+1}^F$" by a formula of L (in a first order way over $M_{\sigma+1}(F)$) and by (1) and (2) this can be done effectively.

Let $f(<2^e,a>)$ be a Gödel number for such a formula.

Now $\mathcal{O}_{\sigma+1}^F = \{<n,b'> \varepsilon I \mid <n,b'> \varepsilon \mathcal{O}^F \;\&\; |<n,b'>|^F \leq \sigma\}$

$\therefore$ $b \varepsilon \mathcal{O}_{\sigma+1}^F \iff [b \varepsilon \mathcal{O}_\sigma^F] \vee [b \varepsilon \mathcal{O}^F \;\&\; |b|^F = \sigma]$

1) by induction we know that $\mathcal{O}_\sigma^F$ is first order definable over $M_\sigma(F)$. It is thus enough to show that (as $<2^e,a>$ is an h-index for $\sigma+1$) the relation $[b' \varepsilon \mathcal{O}^F \;\&\; |b'|^F = \sigma]$ can be expressed in a first order way (over $M_{\sigma+1}(F)$)

2) But $[b' \varepsilon \mathcal{O}^F \;\&\; |b'|^F = \sigma]$ means "b' looks like a notation but $b' \notin \mathcal{O}_\sigma^F$ and b' is of the form $<2^c,a'>$ with $<c,a'>$ a notation in $\mathcal{O}_\sigma^F$ or b' is of the form

$<3^c \cdot 5^m, a'>$ with $<m,a'> \varepsilon\ O^F_\sigma$ and $W_e^{H^F|<m,a'>|} \subseteq O^F_\sigma$ and σ is a limit ordinal or $b' = <1,->$."

i.e. $(\exists k\ \varepsilon\ \omega)(\exists d\ \varepsilon\ I)[b' = <k,d>\ \&\ b' \notin O^F_\sigma$

$\&\ ([k=1] \vee [(\exists m\ \varepsilon\ \omega)(k = 2^m\ \&\ <m,d>\ \varepsilon\ O^F_\sigma)] \vee [(\exists m,\ell\ \varepsilon\ \omega)$

$(k = 3^m \cdot 5^\ell\ \&\ <m,d>\ \varepsilon\ O^F_\sigma\ \&\ W_\ell^{H^F|<m,d>|^F} \subseteq O^F_\sigma$

$\&\ (\forall p\ \varepsilon\ \omega)(\forall d'\ \varepsilon\ I)(<p,d'>\ \varepsilon\ O^F_\sigma \Rightarrow <2^p,d'>\ \varepsilon\ O^F_\sigma))])$

Thus we get effectively (because we know already that we have first order definition for O^F_σ, $W_\ell^{H^F|<m,d>|^F}$, ...) a formula of L and hence we let $g(<2^e,a>)$ be a Gödel number for it

Remark

We assume that we know the rules of formation of notations for ordinals and then we say: either $b\ \varepsilon\ O^F_\sigma$ or $b \notin O^F_\sigma$: in the first case, as we have an h-index for σ, we use the induction hypothesis and in the second case, and if b looks like an index (b is of the form $<1,\ldots>$ or $<2^i,\ldots>$ or $<3^c \cdot 5^m,\ldots>$), then one of the four following cases can arise:

(1) $b = <1,b'>$, then $b\ \varepsilon\ O^F\ \&\ |b|^F = 0$, $\therefore b\ \varepsilon\ O^F_1$

(2) $b = \langle 2^m, b' \rangle$ and we know that $\langle m,b' \rangle \in \mathcal{O}^F_\sigma$,

$\therefore$ $b \in \mathcal{O}^F_{\sigma+1}$

(3) $b = \langle 3^e \cdot 5^m, b' \rangle$ and we know that

(i) $\langle m,b' \rangle \in \mathcal{O}^F_\sigma$ (say $|\langle m,b \rangle|^F = \alpha$)

and (ii) $W_e^{H^F_\alpha} \subseteq \mathcal{O}^F_\sigma$

$\therefore$ b is a notation for a limit ordinal

and (iii) σ is a limit ordinal.

by $b \notin \mathcal{O}^F_\sigma$, we have $|b|^F = \lambda \geq \sigma$

by (i) and (ii), we have $|b|^F = \lambda \leq \sigma$

$\therefore$ $|b|^F = \sigma$ and $b \in \mathcal{O}^F_{\sigma+1}$

(3) otherwise: b is not a notation for σ

Case 2 Let $\langle 3^e \cdot 5^m, a \rangle$ be an h-index for λ, λ limit.

$\therefore$ $\tau = |\langle e,a \rangle|^F < \lambda$

$H^F_\lambda = \{\langle b,c \rangle \mid b \in \mathcal{O}^F_\lambda \;\&\; c \in H^F_{|b|}\}$

We have (1) $b' \in \mathcal{O}^F_\lambda \iff b' \in \mathcal{O}^F \;\&\; |b'|^F < \lambda$

$\iff (\exists c)(\exists d)(c \in (W_m^{H^F_\tau} \cup \{\langle e,a \rangle\})$

$\&$ d is a finite successor of c $\&$ $b' \in \mathcal{O}^F_{|d|^F})$

where "d is a finite successor of c" is defined as in case 2, lemma 0.2.

This works by induction hypothesis: as $<3^e \cdot 5^m, a>$ is an h-index for λ, λ limit we know that

(1) $<e,a> \in \mathcal{O}^F$ & $|<e,a>|^F < \lambda$

(2) $W_m^{H|<e,a>|^F} \subseteq \mathcal{O}^F$

where $W_m^{H|<e,a>|^F}$ is first order definable over $M_{|<e,a>|^F}(F)$. So this gives us a formula defining $\mathcal{O}_\lambda^F$ over $M_\lambda(F)$.

Let $g(<3^e \cdot 5^m, a>)$ be a Gödel number for that formula.

(2) $b' \in H_\lambda^F \iff (\exists b_0, b_1 \in I)(b_0 \in \mathcal{O}_\lambda^F$ & $b_1 \in H^F_{|b_0|^F}$

$$\& \; b' = <b_0, b_1>$$

by (1), $b_0 \in \mathcal{O}_\lambda^F$ is first order definable (effectively) and by induction hypothesis <u>and</u> the fact taht b_0 is now known to be in $\mathcal{O}_\lambda^F$, the relation $b_1 \in H^F_{|b_0|^F}$ is first order definable (effectively) over $M_\lambda(F)$.

This gives a formula defining H_λ^F. Let $f(<3^e \cdot 5^m, a>)$ be a Gödel number for this formula.

$\dashv$

Before we leave the wonderful frame of h-recursion, we want to point out a few consequences of propositions 1.1 and 1.2 (and of the properties of the h-recursive functionals)

Corollary 1.3 (Harrington)

There is a 1-1 recursive correspondence $e \longleftrightarrow \phi_e$ between integers e and Σ_1 sentences ϕ_e in L such that, for all $G \in Tp(n+2)$

$$\{e\}^{\langle F,G\rangle}(0) \downarrow \text{ iff } M_{\kappa_0^{\langle F,G\rangle}}(\langle F,G\rangle) \models \phi_e$$

(see definition 1.1)

Corollary 1.4

(i) $M_{\kappa^F}(F)$ is not admissible

(ii) $M_{\kappa^F_{n-1}}(F)$ is admissible with gaps i.e. the part of $M_{\kappa^F_{n-1}}(F)$ which uses subindividuals as notations for ordinals is an admissible structure.

For the last part of this chapter we assume that $n(\geq 1)$ is a fixed integer and that $F = \langle F', {}^{n+2}E\rangle$ where F' is some fixed object of type n+2.

Proposition 1.5

h-recursive => s-recursive

Proof: it is enough to show that there are s-recursive functions f and g such that

If $\langle k,a\rangle$ is an h-index for σ

Then $f(\langle k,a\rangle)$ and $g(\langle k,a\rangle)$ are defined and

(i) H^F_σ is s-recursive in $\langle S^F_\sigma, {}^{n+2}E, a\rangle$ via Gödel number $f(\langle k,a\rangle)$

(ii) O^F_σ is s-recursive in $\langle S^F_\sigma, {}^{n+2}E, a\rangle$ via Gödel number $g(\langle k,a\rangle)$

The proof is by induction on $|\langle k,a\rangle|^F$

Case 0 $\langle 1,a\rangle$ is an h-index for 0

Let e_0 be the smallest Gödel number for $\emptyset$

Let $f(\langle 1,a\rangle) = g(\langle 1,a\rangle) = e_0$

Case 1 $\langle 2^e,a\rangle$ is an h-index for $\sigma+1$

$\therefore$ $\langle e,a\rangle$ is an h-index for σ

Let $f(\langle e,a\rangle) = e_1$ and $g(\langle e,a\rangle) = e_2$

$\therefore$ H^F_σ is s-recursive in $\langle S^F_\sigma, {}^{n+2}E, a\rangle$ via e_1

But $\langle i,b,x\rangle \in H^F_{\sigma+1} \iff [x = 0 \;\&\; b \in W_i^{H^F_\sigma}] \vee$

$$[x > 0 \;\&\; F(\{i\}_p^{\langle H^F_\sigma, b\rangle}) = x]$$

$\therefore$ $H^F_{\sigma+1}$ is s-recursive in $<S^F_{\sigma+1}, {}^{n+2}E, a>$ via some Gödel number e_3

Let $f(<2^e, a>) = e_3$

Now $\mathcal{O}^F_{\sigma+1} = \{b \in \mathcal{O}^F_\sigma\}$ $\{b \in \mathcal{O}^F \mid |b|^F = \sigma\}$

By an argument similar to that used in proposition 1.2, we can show that $\mathcal{O}^F_{\sigma+1}$ is recursive in $<S^F_{\sigma+1}, {}^{n+2}E, a>$ via some Gödel number e_4.

Let $g(<2^e, a>) = e_4$

<u>Case 2</u> $<3^m \cdot 5^e, a>$ is an h-index for λ, λ limit.

$\therefore$ $<m, a>$ is an h-index for some $\sigma < \lambda$

and: $W_e^{<H^F_\sigma, a>} \subseteq \mathcal{O}^F$ and is s-recursive in $<S^F_{\sigma+1}, {}^{n+2}E, a>$,

thus in $<S^F_\lambda, {}^{n+2}E, a>$ let $f(<m, a>) = e_1$ and $g(<m, a>) = e_2$

We have:

(i) $b' \in \mathcal{O}^F_\lambda \iff b' \in \mathcal{O}^F \;\&\; |b'|^F < \lambda$

as in proposition 1.2, case 3, we can prove that $\mathcal{O}_\lambda$ is s-recursive in $<S^F_\lambda, {}^{n+2}E, a>$ via some Gödel number e_4.

Let $g(<3^m \cdot 5^e,a>) = e_4$

(ii) $b' \varepsilon H^F_\lambda \iff (\exists b_0,b_1)(b_0 \varepsilon O^F_\lambda \ \&\ b_1 \varepsilon H^F_{|b_0|^F} \ \&\ b' = <b_1,b_0>)$

as in proposition 1.2, we can prove that H^F_λ is s-recursive in $<S^F_\lambda, {}^{n+2}E,a>$ via some Gödel number e_3.

Let $f(<3^m \cdot 5^e,a>) = e_3$

<u>Proposition 1.6</u>

s-recursive => h-recursive

To simplify the notations, assume that

$F = <F', {}^{n+2}E>$ and $F' = <F'',g>$ where F'' is a fixed (type n+2) functional and g is a fixed (type n+1) function.

It is enough to show that there is an h-recursive function f such that

<u>If</u> <n,a> is an s-index for σ

<u>Then</u> f(<n,a>) is defined and S^F_σ is h-primitive recursive, with Gödel number f(<n,a>), in $H^F_{\sigma+1}$ and <n,a>.

<u>Case 0</u> <1,a> is an s-index for 0

$x \varepsilon S^F_0 \iff (\exists y \varepsilon I)[(x = <1,y>) \vee (\exists n \varepsilon \omega)(x = <1,y,n> \ \&\ gy = n)]$

Let $f(<1,a>)$ be a Gödel number of a formula defining S_0^F over $<I,H_1^F>$ (note that we need H_1^F because $H_0^F = \emptyset$ and does not give information about the values of g, while H_1^F gives all indications concerning the values of $g \upharpoonright I$).

<u>Case 1</u> $<2^e,a>$ is an s-index for $\sigma+1$

1) for some $i \in \omega$, $<i,a>$ is an s-index for σ

2) $<2^e,a> \notin S_\sigma^F$

3) there is a function h s-recursive in $<S_\sigma^F, {}^{n+1}E, a>$ via Gödel number e (by induction <u>on the type</u> in the definition of s-recursion). Now $<i,a>$ is an s-index for σ iff $f(<i,a>)$ is defined, say $f(<i,a>) = k$, and k is the Gödel number of a set T h-primitive recursive in $H_{\sigma+1}^F$ & $<2^e,a> \notin T$ & $<i,a> \in T$ & $\{f'(e)\}_h^{<{}^{n+1}E,T,a>}$ is total (where the function f' is obtained from the function used for the previous type) & $T = S^F_{|<i,a>|_s}$.

The first part of this expression gives a definition of "$<i,a>$ is an s-index for τ" if $T = S_\tau^F$ and the last part $(T = S^F_{|<i,a>|_s})$ is h-primitive recursive in $H^F_{\sigma+1}$ (and $<2^e,a>$) because we assume by the first part of this sentence that $<i,a>$ is an s-index.

$\therefore$ $\{i \mid i \in \omega \;\&\; <i,a> \text{ is an s-index for } \sigma\}$ is h-semirecursive in $<F,<2^e,a>>$ (a is fixed) and not empty.

$\therefore$ By Gandy (0,5) we can select effectively an i_0 such that $<i_0,a>$ is an s-index for σ.

$\therefore$ we have now $<i_0,a>$, an s-index such that $|<i_0,a>|_s = \sigma$.

But $X \in S^F_{\sigma+1} \iff [X \in S^F_\sigma] \vee [X$ is an s-index for $\sigma+1] \vee [X = <3^c,d,v,m> \;\&\; <2^c,d>$ is an s-index for $\sigma+1 \;\&\; \{c\}^G_s(b) = m] \vee [X = <5^c,d,m> \;\&\; <2^c,d>$ is an s-index for $\sigma+1 \;\&\; F''(\lambda b \mid \{c\}^G_s(b)) = m]$

where $G = <S^F_\sigma, {}^{n+1}E, d, <2^e,a>>$

$\therefore$ $S^F_{\sigma+1}$ is first order definable over $<I,H^F_{\sigma+2}>$ if "X is an s-index for $\sigma+1$" is also first order definable; but this is obvious (because we start with an s-notation for $\sigma+1$).

Thus the only things that we must say are:

(i) X is of the form $<2^e,a>$ $(e \in \omega, e \neq 0)$

(ii) $X \notin S^F_\sigma$

(iii) $\{e\}_s^{<^{n+1}E,S^F_\sigma,a>}$ is total.

(i) and (ii) are trivial (by induction on the ordinals) and (iii) also (by induction on the type)

Let $f(<2^e,a>)$ be the smallest Gödel number for the formula defining $S^F_{\sigma+1}$ over $<I,H^F_{\sigma+2}>$

Case 2 $<7^e,a>$ is an s-index for λ, λ limit

$\therefore$ $<2^e,a>$ is an s-index for some S^F_σ, $\sigma < \lambda$

$\therefore$ $f(<2^e,a>)$ is defined and $\{e\}^G_s$ is an s-recursive set of s-notations for ordinals (with $G = <S^F_\sigma, {}^{n+1}\Xi, a>$).

By induction on type and ordinals, it is clear that S^F_λ is first order definable over $<I,H^F_{\lambda+1}>$. Let $f(<7^e,a>)$ be the smallest Gödel number for the formula defining S^F_λ over $<I,H^F_{\lambda+1}>$

Remarks

1) Obviously when we say recursive in $\ldots_\sigma$ we usually mean, but sometimes "forget" to write, recursive in $\ldots_\sigma$

and a notation for σ. It is also obvious that most of the proofs we gave here use the recursion theorem, although we carefully "forgot" to mention this fact.

2) We use, in the proof of 1.6, Gandy's theorem (0.5): the proof of this theorem was given in Harrington's frame. We use it in this frame to show that we have a selection theorem for integers (for h-semirecursive sets) and we apply this result to the set $\{i \in \omega \mid \cdots\}$ which we consider in the proof of proposition 1.6 (case 1).

Corollary 1.7

By propositions 1.5 and 1.6, it is now obvious that h-recursion and s-recursion coincide. Hence that Gandy's theorem (0.5) and Grilliot's selection theorem (0.7) hold in both Harrington's settings as well as in the frame of Sacks' definition.

Remark

We did not need to prove proposition 1.5: proposition 1.6 shows that s-recursive => h-recursive; 1.8 will show that h-recursive => k-recursive and 1.9 will show that k-recursive => s-recursive.

Nevertheless, our proof of 1.9 requires the use of Grilliot's result in Sacks' frame, but the proof of Grilliot's result itself is not as easy in that situation as in Harrington's universe. Moreover, this proof (in Sacks' frame) is not as similar to that of the Gandy selection theorem as we could expect it to be. This is why we preferred to prove both theorems inside Harrington's universe; and why we had to use proposition 1.5 to show that all these results were also true in Sacks' frame.

Proposition 1.8

h-recursive => k-recursive

It is enough to prove that for each $a \in \mathcal{O}^F$, $\mathcal{O}^F_{|a|^F}$ and $H^F_{|a|^F}$ are k-recursive in F, uniformly in a. Thus it is enough to show that there are k-recursive functions f and g such that:

if <m,a> is an h-index for σ,

then g(<m,a>) and f(<m,a>) are defined and

$H^F_\sigma = \{f(\langle m,a\rangle)\}^F_k$ and $\mathcal{O}^F_\sigma = \{g(\langle m,a\rangle)\}^F_k$. The proof is by induction on $|a|^F$.

Case 0 <1,a> is an h-index for 0

Let $f(\langle 1,a\rangle) = g(\langle 1,a\rangle) = e_0$ be the least k-index for $\emptyset$

Case 1 $\langle 2^e,a\rangle$ is an h-index for $\sigma+1$

$\therefore$ <e,a> is an h-index for σ

and by induction hypothesis $H_{\sigma}^{F} = \{f(\langle e,a \rangle)\}_{k}^{F}$

(1) $\langle e,y,m \rangle \varepsilon H_{\sigma+1}^{F} \Longleftrightarrow [m = 0 \ \& \ y \ \varepsilon \ W_{e}^{H_{\sigma}^{F}}]$

$\vee [m > 0 \ \& \ F(\{e\}_{p}^{\langle H_{\sigma}^{F},y \rangle}) = m-1]$

$\therefore$ There is a formula describing $H_{\sigma+1}^{F}$ in Kleene's universe. Let $f(\langle 2^{e},a \rangle)$ be the smallest k-index for this set:

(2) $b \ \varepsilon \ \mathcal{O}_{\sigma+1}^{F} \Longleftrightarrow b \ \varepsilon \ \mathcal{O}^{F} \ \& \ |b|^{F} < \sigma+1$

$\Longleftrightarrow [b \ \varepsilon \ \mathcal{O}_{\sigma}^{F}] \vee [b \ \varepsilon \ \mathcal{O}^{F} \ \& \ |b|^{F} = \sigma]$

$\Longleftrightarrow [b \ \varepsilon \ \mathcal{O}_{\sigma}^{F}] \vee [(\exists k \ \varepsilon \ \omega)(\exists d \ \varepsilon \ I)$

$(b = \langle k,d \rangle \ \& \ b \notin \mathcal{O}_{\sigma}^{F} \ \& \ [(k=1) \vee (\exists m \ \varepsilon \ \omega)(k = 2^{m}$

$\& \ \langle m,d \rangle \ \varepsilon \ \mathcal{O}_{\sigma}^{F}) \vee (\exists m,\ell \ \varepsilon \ \omega)(k = 3^{m} \cdot 5^{\ell} \ \& \ \langle m,d \rangle \ \varepsilon \ \mathcal{O}_{\sigma}^{F} \ \&$

$W_{\ell}^{H_{|\langle m,d \rangle|^{F}}^{F}} \subseteq \mathcal{O}_{\sigma}^{F} \ \& \ (\forall p \ \varepsilon \ \omega)(\forall d' \ \varepsilon \ I)(\langle p,d' \rangle \ \varepsilon \ \mathcal{O}_{\sigma}^{F} \Longrightarrow$

$\langle 2^{p},d' \rangle \ \varepsilon \ \mathcal{O}_{\sigma}^{F}))])]$.

As $\mathcal{O}_{\sigma}^{F}$, $H_{|\langle m,d \rangle|^{F}}^{F}$, $W_{\ell}^{H_{|\langle m,d \rangle|^{F}}^{F}}$ have k-indices by induction hypothesis, this gives us again a formula describing, in Kleene's universe, the relation $b \ \varepsilon \ \mathcal{O}_{\sigma+1}^{F}$.

$\therefore$ We can get a k-index for $\mathcal{O}_{\sigma+1}^{F}$.

Let $g(<2^e,a>)$ be the smallest such index.

Case 2 $<3^m \cdot 5^e,a>$ is an h-index for λ, λ limit

$\therefore$ $<m,a>$ is an h-index for some $\sigma < \lambda$ and

$W_e^{<H_\sigma^F,a>} \subseteq \mathcal{O}_\lambda^F$ (with $\sigma = |<m,a>|^F$)

(1) Then $b \in \mathcal{O}_\lambda^F \iff (\exists c)(\exists d)(c \in (W_e^{H_\sigma^F} \cup \{<m,a>\})$

$\&$ d is a finite successor of c $\&$ $b \in \mathcal{O}^F_{|d|^F})$

By induction hypothesis, it is easy to see that this gives us a k-index, e_0, for the relation $b \in \mathcal{O}_\lambda^F$. Let $g(<3^m \cdot 5^e,a>) = e_0$.

(2) $<b,c> \in H_\lambda^F \iff b \in \mathcal{O}_\lambda^F \;\&\; c \in H^F_{|b|}$

By induction hypothesis and (1) this gives us a k-index e_1 for the relation $<b,c> \in H_\lambda^F$. Let $f(<3^m \cdot 5^e,a>) = e_1$

For the next proof we assume that a codes a finite list of objects of finite type, that 2 is the highest type used as argument of Φ (the function being defined), that n_i is the number of objects of type i arguments of Φ and that $(a)_i$ codes the objects of type exactly i, arguments of Φ.

<u>Proposition 1.9</u>

k-recursive => s-recursive

It is enough to define recursive functions f and g such that:

<u>If</u> $(\exists m)(\langle e,F*a,m\rangle \in K)$ (i.e.: if $\{e\}_k^F(a) \downarrow$)

<u>Then</u> $\langle fe,a\rangle$ is an s-index, say $|\langle fe,a\rangle|_s = \sigma$ and

$$\{e\}_k^F(a) \simeq \{ge\}_s^G \text{ with } G = \langle S_\sigma^F, {}^{n+1}E, a\rangle$$

The proof is by induction on the schemas

S1) e is $\langle 1, \langle n_0,\ldots,n_r\rangle\rangle$

Then $$\Phi(a) = \begin{cases} (a)_{0,0}+1 & \text{if } n_0 \neq 0 \\ \uparrow & \text{otherwise} \end{cases}$$

(Note that if $n_0 = 0$, then e is <u>not</u> a k-index for a recursive functional, thus $(\forall m)(\langle e,F*a,m\rangle \notin K)$ and there is nothing to prove).

Let e_0 be a Gödel number (in Sacks' universe) for the following set of instructions: "look at n_0 & if $n_0 \neq 0$, take $(a)_{0,0}$ and add 1".

Let fe = 1 ∴ $\langle fe,a\rangle$ is an s-index for 0

Let $ge = e_0$

Then $\{e\}_k^F(a) \simeq \{e_0\}_s^{\langle S_0^F, {}^{n+1}E, a\rangle}$

S2) e is $\langle 2, \langle n_0,\ldots,n_r\rangle, q\rangle$

Then $\Phi(a) = q$

Let e_1 be a Gödel number for the following set of instructions (in Sacks' universe, of course!): "let $\Phi(a)$ be q".

Let $fe = 1$ $\therefore$ $\langle fe,a\rangle$ is an s-index for 0

Let $ge = e_1$

Then $\{e\}_k^F(a) \simeq \{e_1\}_s^{\langle S_0^F, {}^{n+1}E, a\rangle}$

S3) e is $\langle 3, \langle n_0,\ldots,n_r\rangle\rangle$

Then $\Phi(a) = \begin{cases} (a)_{0,0} & \text{if } n_0 \neq 0 \\ \uparrow & \text{otherwise} \end{cases}$

(Same remark about n_0 as for S1)

Let e_3 be an s-Gödel number for the following set of instructions: "look at n_0 & if $n_0 \neq 0$, then take $(a)_{0,0}$"

Let $fe = 1$ $\therefore$ $\langle fe,a\rangle$ is an s-index for 0

Let $ge = e_3$

Then $\{e\}_k^F(a) \simeq \{e_3\}^{\langle S_0^F, {}^{n+1}E, a\rangle}$

S4) e is $\langle 4, \langle n_0,\ldots,n_r\rangle, u, v\rangle$

<u>Then</u> $\{e\}_k^F(a) \simeq \{u\}_k^F(\{v\}_k^F(a),a)$

Let $b = \langle\{v\}_k^F(a),a\rangle$

By induction hypothesis, $\langle fu,b\rangle$ is an s-index, say $|\langle fu,b\rangle|_s = \sigma$, and $\{u\}_k^F(b) \simeq \{gu\}_s^{G(\sigma)}$ with

$G(\sigma) = \langle S_\sigma^F, {}^{n+1}E, b\rangle$ and also $\langle fv,a\rangle$ is an s-index, say $|\langle fv,a\rangle|_s = \tau$ and $\{v\}_k^F(a) \simeq \{gv\}_s^{G(\tau)}$ with

$G(\tau) = \langle S_\tau^F, {}^{n+1}E, a\rangle$.

<u>So</u> compute first $x = \{gv\}_s^{G(\tau)}$

<u>Then</u>, the set $\{\langle fv,a\rangle, \langle fu,x*a\rangle\}$ is s-recursive in

$S_\tau^F = S^F_{|\langle fv,a\rangle|_s}$ (note that by induction hypothesis $\langle fu,b\rangle$ is an s-index, but $\langle fv,a\rangle$ is "big enough" to compute $x = \{gv\}_s^{G(\tau)}$ and $\langle fu,b\rangle = \langle fu,x*a\rangle$!)

Expand the set $\{\langle fv,a\rangle, \langle fu,b\rangle\}$ by taking all successors. This gives us a new set T from which we can get an s-index $\langle 7^{e^*},a\rangle$ for the first limit ordinal strictly bigger than max $\{\tau,\sigma\}$ where we get e^* effectively from e. (i.e. we expand the first set of ordinals to another one having a limit as least upper bound; this "expansion part" would not be needed if we were still in Harrington's setting).

Let $fe = 7^{e^*}$ $\therefore$ $<fe,a>$ is an s-index.

Let then ge be an s-Gödel number for the following set of instructions: "Get $\tau = |<fv,a>|_s$ and S^F_τ from $S^F_{|<7^{e^*},a>|_s}$, compute $x = \{gv\}_s^{G(\tau)}$ with $G(\tau) = <S^F_\tau, {}^{n+1}E, a>$

get $\sigma = |<fu,x*a>|_s$ and S^F_σ from $S^F_{|<7^{e^*},a>|_s}$ and

compute $\{gu\}_s^{G(\sigma)}$ with $G(\sigma) = <S^F_{|<fu,b>|_s}, {}^{n+1}E, b>$

and $b = x*a$.

Then $\{e\}^F_k(a) \simeq \{ge\}_s^{<S^F_{|<fe,a>|_s}, {}^{n+1}E, a>}$.

S5) It is possible to give a direct proof in this case, but as we will later prove that proposition 1.9 holds for scheme S9 we do not need S5. Hence we will not give a proof for this case.

S6) e is $<6, <n_0,\ldots,n_r>, j, k, u>$

<u>Then</u> $\{e\}^F_k(a) = \{u\}^F_k(\tilde{a})$

where $\tilde{a}$ is obtained from a by moving the $k+1^{\underline{st}}$ type j argument in front of the list of type j arguments (we assume $n_j \geq k+1$)

Let $fe = (fu)^*$ where * means: "go to the successor"

$\therefore$ by induction hypothesis $<fe,a>$ is an s-index say $|<fe,a>|_s = \sigma+1 = |<fu,a>|_s+1$.

Let u' be an s-Gödel number such that u' describes the permutation giving $\tilde{a}$ from a and carries all the information contained in gu.

Let $ge = u'$.

Then $\{e\}_k^F(a) \simeq \{ge\}_s^{\langle S_{\sigma+1}^F, {}^{n+1}E, a\rangle}$

S7) e is $\langle 7, \langle n_0,\ldots,n_r\rangle\rangle$

Then $\Phi(a) = \begin{cases} (a)_{1,0}((a)_{0,0}) & \text{if } n_0 > 0 \;\&\; n_1 > 0 \\ \uparrow & \text{otherwise.} \end{cases}$

(Note that if $n_0 = 0$ or $n_1 = 0$, then e would not be a k-index and there would be nothing to prove).

Let e' be an s-Gödel number for the following set of instructions: "look at n_0 and n_1, and if both are different of 0, take $\alpha = (a)_{1,0}$ and $k = (a)_{0,0}$ and compute $\alpha(k)$"

Let $fe = 1$ $\therefore$ $\langle fe,a\rangle$ is an s-index

Let $ge = e'$.

Then $\{e\}_k^F(a) \simeq \{ge\}_s^{\langle S^F_{|\langle fe,a\rangle|}, {}^{n+1}E, a\rangle}$

S8) e is $\langle 8, \langle n_0,\ldots,n_r\rangle, j, u\rangle$

Then $\Phi(a) = \begin{cases} (a)_{j,0}(\lambda\beta^{j-2} \mid \chi((a)_{j,0},\beta^{j-2},a) \\ \uparrow \quad \text{otherwise} \end{cases}$

where χ is $\{u\}_k^F$.

(Note that $\Phi(a)$ diverges if $\{u\}_k^F$ is not total; same remark as previously about n_j)

Part 1 $j < n+2$

Then the result is a trivial consequence of the closure properties of the relations semirecursive in F under quantification (both existential and universal) over $Tp(n-1)$.

i.e. $\Phi(\underline{a}^j) \simeq k \iff (\exists c \in Tp(j-1))(c = \lambda b^{j-2} \mid \chi(\underline{a},b)$

$\& \ (\underline{a})_0^j(c) \simeq k)$

where $\underline{a}^j$ codes a finite sequence as one type j object and $(\underline{a})_0^j$ is the first object of type j of that sequence.

Let fe and ge be the appropriate integers.

This proof, however, does not work for $j = n+2$, because semirecursive in F relations are not closed under existential quantification over $Tp(n) = I$:

Part 2 $j = n+2$

Then

$$\{e\}_k^F(a) \simeq F(\lambda\beta^n \mid \{u\}_k^F(\beta^n,a))$$

Thus by induction hypothesis:

$\{\langle fu,\beta^n * a\rangle \mid \beta^n \in I\}$ is a recursive set of s-notations for ordinals; as previously we expand this set to a limit and get an s-notation $\langle 7^{e^*},a\rangle$ (where e^* has been obtained effectively from e).

Let $fe = 7^{e^*}$ ∴ $\langle fe,a\rangle$ is an s-index say $|\langle 7^{e^*},a\rangle|_s = \lambda$.

Let ge be an s-Gödel number for the following set of instructions:

"1) compute from λ (and $\langle fe,a\rangle$) the ordinal $\sigma < \lambda$ such that S_σ^F gives us the value of

$$F(\lambda\beta^n \mid \{gu\}_s^{G(\beta)})$$

where $G(\beta) = \langle S^F_{|\langle fu,\beta^n * a\rangle|_s}, {}^{n+1}E, \beta^n * a\rangle$ and

2) take the value given there."

Then $\{e\}_k^F(a) \simeq \{ge\}_s^G$

where $G = \langle S^F_{|\langle fe,a\rangle|_s}, {}^{n+1}E, a\rangle$

S9) e is $<9, <n_0,\ldots,n_{r_0}>, <m_0,\ldots,m_{r_1}>>$

Then $\Phi(x,b,c) \simeq \{x\}_k^F(b)$

(We assume here that a is of the form $<x,b,c>$, and that $n_0 \neq 0$ ... otherwise, there is nothing to prove.)

∴ $\{e\}_k^F(x,b,c) \simeq \{x\}_k^F(b)$

∴ by induction hypothesis fx is defined and

$<fx,b>$ is an s-index, say $|<fx,b>|_s = \tau$

Let $fe = (fx)^*$ where $*$ means: "if we want to compute the ordinal $|<fe,x*b*c>|$, then we first forget the x and c parts of $x*b*c$, we compute the ordinal $|<fx,b>|$ and finally we get an s-index for its successor."

∴ $<fe,x*b*c>$ is an s-index, namely for

$|<fx,b>|_s+1 = \tau+1$.

Let ge be an s-Gödel number for the following set of instructions: "get x and b from $<x,b,c>$; compute from $\tau+1$ the value of $\{gx\}_s^{<S_\tau^F, {}^{n+1}E, b>}$ and assign it as value of Φ at $<x,b,c>$"

Then

$\{e\}_k^F(x,b,c) \simeq \{ge\}^{<S_{\tau+1}^F, {}^{n+1}E, x*b*c>}$ ⊣

It is obvious that all the definitions of recursive functionals that we have considered here are generalizations of the Ordinary Recursion Theory: the easiest way to notice this fact is probably to look at the definition of "k-recursive in".

2. A minimal pair of individuals

The problem of the minimal pair has first been solved (independently) by Lachlan [12] and Yates [32] in Ordinary Recursion Theory. It can be stated as follows: "Is it possible to find two semirecursive sets of integers, A and B, such that neither is recursive in the other one and such that every real D recursive in both A and B is recursive?" The answer to this question is yes. Lachlan and Yates use both, somewhere in their proof, a priority argument. This priority argument is more complicated than the usual "Post-problem-type" argument: to solve Post problem, one must consider at each step of the construction one candidate for membership in A (or B) and then make a decision. This cannot be the case in a minimal pair construction: in this construction we want to preserve equalities of the form $\{e\}^A = \{f\}^B$ as far as possible, i.e. on an initial segment of ω as long as possible. Hence we must consider at each step a set of candidates for A (or B) and when we "think" that the equalities mentioned above have been satisfied on an initial segment of ω of maximal length, then, but not earlier, we commit ourselves to put (possibly) one of these candidates in A (or B). The main advantage of Ordinary Recursion Theory is that in this case, everything is finite. If we try to generalize these proofs to α-recursion theory, for every Σ_1-admissible α, we might have problems because of the existence of (many) limit ordinals $\lambda < \alpha$. Suckonick made a first step towards the solution of this problem: he constructed [30] two meta-r.e. subsets of ω_1^{CK} such that none of them was meta-recursive in the other one, but such that everything meta-recursive in both was meta-recursive. Lerman and Sacks [14] gave later a proof of the existence of a minimal pair of α-r.e. sets for each Σ_1-admissible α, provided that α is not refractory; i.e.: provided that the following is false: $p2\alpha = gc\alpha < tp2\alpha \leq \alpha$. (Where $p2\alpha$ is the Σ_2 projectum of α, $gc\alpha$ is the greatest cardinal of L_α (if there is one and α otherwise) and $tp2\alpha$ is the tame Σ_2 projectum of α).

The solution of this problem seems thus to exist in α-recursion theory. Unluckily the main ordinal we get when we do recursion in a normal object F of type n+2 (F and n fixed), κ^F, is not admissible as soon as $n > 0$. It is easy however to see that λ_{n-1}^F has then all the nice properties we expect to find in a Σ_1-admissible ordinal. We will use this fact to construct a minimal pair of sets of subindividually, semirecursive in F. Another useful fact is the following: there are only countably many functionals recursive in F: this is of course easier than in α-recursion theory. We will work in the frame of Kleene's definition and use reducibilities defined by MacQueen in [15]. This will allow us to give a proof for every n (n > 0).

From now on we assume that $n \geq 1$ is a fixed integer and that F is a fixed normal type n+2 object (e.g. $F = \langle F', {}^{n+2}E \rangle$ where F' is a fixed object of type n+2).

Let S be the set of all the ordinals $\sigma < \kappa^F_{n-1}$ such that σ has a notation from a subindividual. $\kappa^F_{n-1} = \sup S$ and $\lambda^F_{n-1} =$ order type of S. Let $t : \lambda^F_{n-1} \xrightarrow[\text{into}]{1-1} \kappa^F_{n-1}$ be the unique order preserving isomorphism between λ^F_{n-1} and S.

Our construction will be based on type n-1 notations (in fact when we mention an ordinal $\sigma < \lambda^F_{n-1}$, we actually mean the set of all subindividual notations for $t(\sigma)$).

<u>Convention</u>

We write α for λ^F_{n-1}

<u>Definition 2.1</u>

A subindividual a is called a <u>notation from F</u>, and we write $Not_{n-1}(F,a)$ iff

$\{(a)_{0,0}\}[(a)_1{*}F]\downarrow$ iff $(\exists k\varepsilon\omega)(<(a)_{0,0},(a)_1{*}F,k> \varepsilon\ K)$

<u>If</u> $Not_{n-1}(F,a)$, <u>then</u> we write

$|a|_c = |\{(a)_{0,0}\}[(a)_1{*}F]|$ = the length of the computation

$\{|a|_c \mid Not_{n-1}(F,a)\} = S$

(Remember: we showed in the frame of Harrington's definition that $M_{\kappa^F_{n-1}}(F)$ was an admissible set with gaps; we collapse now this structure to get $L_\alpha(F)$ which is admissible.)

By convention $|a|_c = \infty$ if a is not a notation.

By Gandy's theorem (0.5) we can compare two notations and hence we can define $|a|$ to be the height of the prewellordering defined by this comparison on the set $\{b \varepsilon SI \mid |b|_c \leq |a|_c\}$. In other words $|a| = t^{-1}(|a|_c)$ and

$\alpha = \sup \{|a| \mid Not_{n-1}(F,a)\}$

Let $Not_{n-1}(F) = \{a \in SI \mid Not_{n-1}(F,a)\}$

<u>If</u> $Not_{n-1}(F,a)$, <u>then</u> <u>level (a)</u> denotes the set $\{b \in SI \mid |b| = |a|\}$. Any set of this form is called a level. It is easy to see that any level is recursive in F and any of its members, uniformly (use our Fact G in the proof and compare this fact with the numerous uses we made of "0^F_σ is recursive in ..." in chapter 1).

<u>If</u> $\sigma < \alpha$, <u>then</u> <u>level (σ)</u> denotes the set $\{a \in SI \mid |a| = \sigma\}$. In the construction we will mostly use (tacitly) levels.

<u>Definition 2.2</u>

A set $A \subseteq \alpha$ is said to be <u>F-finite</u> (<u>F-subfinite</u> respectively) iff there is a set $S_A \subseteq Not_{n-1}(F)$ such that

1) S_A is recursive in F (recursive in F and a subindividual)

2) $A = \{|a| \mid a \in S_A\}$

An index for A is then a number e (a subindividual $\langle e,a \rangle$) such that $\lambda b^{n-1} \mid \{e\}^F(b)$ $(\lambda b^{n-1} \mid \{e\}^F(a*b))$ is the characteristic function of S_A

A set $A \subseteq \alpha$ is said to be semirecursive (semirecursive in a subindividual) in F iff there is a set $S_A \subseteq Not_{n-1}(F)$ such that

1) S_A is semirecursive in F (semirecursive in F and a subindividual)

2) $A = \{|a| \mid a \in S_A\}$

An index for A (as a semirecursive set) is then defined in the obvious manner.

Remark

We can always assume, without loss of generality, that S_A is well defined with respect to levels.

Definition 2.3

An operator $A : \alpha \to 2^{\alpha}$ taking ordinals $\sigma < \alpha$ to subsets of α is said to be F-partial recursive iff there is an index e_A such that for all $a \in SI$, $\langle e_A, a \rangle$ is an index for $A(\sigma)$ whenever $|a| = \sigma$.

A partial function $f : \alpha \to \alpha$ is said to be F-partial recursive iff there is an index e_f such that for all $a \in SI$, $f(\sigma) \simeq \tau$ iff $\langle e_f, a \rangle$ is an index for $\{\tau\}$ whenever $|a| = \sigma$ iff $\lambda b^{n-1} \mid \{e_f\}^F(a*b)$

is the characteristic function of level τ whenever $|a| = \sigma$.

This shows clearly that we are <u>not</u> interested in all the α-recursive functions but only in those functions <u>which can be indexed by integers</u>.

If A is an operator, then $A^{\sigma} = \{A(\tau) \mid \tau < \sigma\}$.

Let A^{*} be the operator such that $A^{*}(\sigma) = A^{\sigma}$.

We will need now a few facts, some of them have been introduced by MacQueen [15] when he defined his notion of reducibilities.

<u>Fact 2.1</u>

If A is an F-partial recursive operator, so is A^{*} $\dashv$

<u>Fact 2.2</u>

<u>If</u> A is an F-partial recursive operator defined on all of α, <u>then</u> $A = \bigcup\{A(\sigma) \mid \sigma < \alpha\}$ is semirecursive in F.

<u>Proof</u>

Let e_A be an index for A as an F-partial recursive operator. Then define S_A by:

$$b \in S_A \iff b \in SI \;\&\; (\exists a \in SI)(Not_{n-1}(F,a) \;\&\; \{e_A\}^F(a*b) \simeq 0)$$

$$\iff b \in SI \;\&\; (\exists a \in SI)(\exists k \in \omega)(<(a)_{0,0},F*a,k> \in K$$

$$\& <e_A,F*a*b,0> \in K)$$

S_A is clearly semirecursive in F (by Grilliot) and $A = \bigcup\{A(\sigma) \mid \sigma < \alpha\} = \{|b| \mid b \in S_A\}$ is semi-recursive in F. ⊣

<u>Fact 2.3 (Boundedness)</u>

<u>If</u> $B \subseteq \alpha$ is F-finite and A is an F-partial recursive operator defined on all of α, such that $B \subseteq \{A(\sigma) \mid \sigma < \alpha\}$.

<u>Then</u> $B \subseteq A^\sigma$ for some $\sigma < \alpha$.

<u>Proof</u>

Let e_A be an index for A as an F-partial recursive operator.

For every $a \in SI \cap S_B$, define the set S_B^a of all subindividual notations for the first level at which a is put in S_A as follows:

$$b \in S_B^a \iff b \in SI \;\&\; Not_{n-1}(F,b) \;\&\; \{e_A\}^F(b*a) \simeq 0$$

$$\& \; (\forall c \in SI)(|c| < |b| \Rightarrow \{e_A\}^F(c*a) \simeq 1)$$

For every subindividual $a \in S_B$, S_B^a is not empty and semirecursive in $\langle F, a \rangle$ uniformly in a.

Hence (by Grilliot) we can find a recursive in F subset of S_B^a uniformly in a. Then (after filling the gaps in the levels, if needed) we have an index e such that for every $a \in S_B$,

$\lambda b^{n-1} \mid \{e\}^F(a*b)$ is the characteristic function of level (σ_a) where $\sigma_a = \mu\sigma[|a| \in A(\sigma)]$.

Let f be F-partial recursive function with index e. Then f is defined on B and $(\tau \in B \Rightarrow f\tau = \mu\sigma[\tau \in A(\sigma)])$.

<u>If</u> $f \upharpoonright B$ were unbounded in α, <u>then</u> we could define $Not_{n-1}(F)$ recursively in F by:

$$a \in Not_{n-1}(F) \Leftrightarrow a \in SI \;\&\; (\exists b \in SI)(b \in S_B \;\&$$
$$(\exists c \in SI)(\{e\}^F(b*c) \simeq 0 \;\&\; |a| \leq |c|))$$

But $Not_{n-1}(F)$ is not recursive in F (by the usual diagonal argument). Hence $f \upharpoonright B$ must be bounded below α. This implies that $B \subseteq A^\sigma$ for some $\sigma < \alpha$. ⊣

Definition 2.4

$\text{Ind}(a) \iff$ a is an index for a set $A \subseteq \alpha$

F-finite in a subindividual (namely $(a)_1$)

$\iff a \in SI \;\&\; \lambda b^{n-1} \mid \{(a)_{0,0}\}^F((a)_1 * b)$ is total

$\& \; \{b \in SI \mid \{(a)_{0,0}\}^F((a)_1 * b) \simeq 0\} \subseteq \text{Not}_{n-1}(F)$

If Ind (a) holds,

Then let $K_a = \{|b| \mid \{(a)_{0,0}\}^F((a)_1 * b) \simeq 0\}$

i.e. K_a is the subset of α having index a.

Fact 2.4

(i) The predicate Ind is semirecursive in F

(ii) $B \subseteq \alpha$ is F-subfinite iff there is a subindividual a such that Ind (a) holds and $B = K_a$.

Proof

(i) SI is a set recursive in F and $\text{Not}_{n-1}(F)$ is semirecursive in F

$(a \in \text{Not}_{n-1}(F) \iff a \in SI \;\&\; (\exists k \in \omega)(\langle (a)_{0,0}, (a)_1 * F, k\rangle \in K))$

Now $\lambda b^{n-1} \mid \{(a)_{0,0}\}^F((a)_1 * b)$ is total iff

$(\forall b \in SI)(\exists k \in \omega)(\langle (a)_{0,0}, (a)_1 * b * F, k\rangle \in K)$

$\therefore$ this part of the definition is also semirecursive in F and under this assumption

$B = \{b \in SI \mid \{(a)_{0,0}\}^F((a)_1 * b) = 0\}$ is recursive in F.

$\therefore$ Ind (a) is semirecursive in F.

(ii) The proof of part (i) gave us: $K_a = \{|b| \mid b \in B\}$ where B is recursive in a subindividual and F. Thus K_a is F-subfinite.

Assume B' is F-subfinite. Then $B' = \{|b| \mid b \in \tilde{B}\}$ where $\tilde{B}$ is recursive in F and a subindividual. Hence it is easy to get an index for B'. $\dashv$

Thus, the predicate Ind gives us a way to get, as in usual α-recursion theory, all the "finite" subsets of α. The temptation to use these "finite" sets in neighbourhood conditions begins to become irresistible. The following facts will increase this temptation.

<u>Fact 2.5</u>

<u>If</u> e is the integer such that Ind (a) $\Leftrightarrow \{e\}^F(a)\downarrow$
<u>Then</u> (i) Ind (a) $\Rightarrow \langle e,a\rangle \in Not_{n-1}(F)$

(ii) Ind (a) $\Rightarrow \sup\{\sigma \mid \sigma \in K_a\} \leq |\langle e,a\rangle|$

<u>Proof</u>

Assume that Ind (a) $\Leftrightarrow \{e\}^F(a)\downarrow$

Then, to check whether a subindividual $b \in \text{Ind}$, we must check two things:

1) Is $\lambda c^{n-1} \mid \{(b)_{0,0}\}^F((b)_1 * c)$ total?

and 2) Is $C = \{c \in SI \mid \{(b)_{0,0}\}^F((b)_1 * c) \simeq 0\} \subseteq \text{Not}_{n-1}(F)$?

This involves carrying out each of the computations $\{(c)_{0,0}\}^F*((c)_1)$ (for $c \in C$) to check whether there is an integer k such that $\langle (c)_{0,0}, (c)_1 * F, k\rangle \in K$.

Hence assuming that Ind (a) holds, we have a computation which verifies this (hence $\langle e,a\rangle \in \text{Not}_{n-1}(F)$) and this computation must clearly contain subcomputations of height $|b|_c$ for each b such that $|b| \in K_a$. ⊣

Definition 2.5 (Reduction procedures)

For each $e \in \omega$, let $R_e(a,b,c,k)$ be the predicate which holds iff

(i) $a,b,c \in SI$ & $k \in \omega$

(ii) $c \in \text{Not}_{n-1}(F)$

(iii) Ind (a) & Ind (b) & $K_a \cap K_b = \emptyset$ and

(iv) $\{e\}^F(a*b*c*k) \downarrow$

Then R_e is semirecursive in F, uniformly in e

∴ there exists a recursive function $g : \omega \to \omega$ such that $R_e(a,b,c,k) \iff \{ge\}^F(a*b*c*k) \downarrow$

_If_ $\sigma < \alpha$, _then_ the relation $R_e^\sigma(a,b,c,k)$ is defined as follows:

$$R_e^\sigma(a,b,c,k) \iff |\{ge\}^F(a*b*c*k)| < t(\sigma)$$

where $t : \alpha \to \kappa_{n-1}^F$ is the natural injection defined at the beginning of this chapter (with $t(\sigma) = |d|_c$ for any $d \in SI \cap \text{level}(\sigma)$).

$R_e^\sigma(a,b,c,k)$ is clearly recursive in $\langle F,d\rangle$ for any $d \in \text{level}(\sigma)$, uniformly.

_If_ $A \subseteq \alpha$, _then_ we define a partial function $[e]^A : \alpha \to \omega$ by:

$$[e]^A(\sigma) \simeq y \iff (\exists a,b,c \in SI)[R_e(a,b,c,y)$$
$$\& \ K_a \subseteq A \ \& \ K_b \subseteq \alpha - A \ \& \ |c| = \sigma]$$

For any $\tau < \alpha$ we define similarly:

$$[e]_\tau^A(\sigma) \simeq y \iff (\exists a,b,c \in SI)[R_e^\tau(a,b,c,y)$$
$$\& \ K_a \subseteq A \ \& \ K_b \subseteq \alpha - A \ \& \ |c| = \sigma]$$

Remark

In general $[e]^A$ and $[e]^A_\tau$ are many valued.

Definition 2.6 (Weakly recursive in)

Let $A, B \subseteq \alpha$

We say that B is weakly recursive in A and write $B \leq_{wr} A$ iff there exists an integer e such that $B = [e]^A$. B is said to be recursive iff $B = [e]$ for some e.

Fact 2.6

If A is F-finite, then the relation $[e]^A_\tau(\sigma) \simeq y$ is recursive in $\langle F, a, b\rangle$ where a and b are respectively notations for σ and τ, uniformly.

Fact 2.7

If $R^\sigma_e(a,b,c,y)$ holds

Then $K_a \subseteq \sigma$ and $K_b \subseteq \sigma$

Proof

This is an immediate consequence of fact 2.5

We are now ready to state (and prove) the main result of this section:

Theorem 2.8

Let $n > 0$. There are two sets $A, B \subseteq \alpha$, F-semirecursive, such that A is not recursive and B is not recursive and such that any set recursive in both is recursive.

The proof requires a priority argument. We shall construct sets A and B by stages. We need α stages. By fact 2.2 these sets will be F-semirecursive.

We will try to save as long as possible equalities of the form $[e]^A = [f]^B$ $(e, f \in \omega)$ and we will destroy them only when we cannot resist the temptation. There will be only countably many requirements. We will use this fact to be able to well order the requirements in advance. The construction will be such that $[e]^A$, $[e]^B$ will be single valued for all $e \in \omega$ (A,B will be "regular").

Definition 2.6'

There will be positive requirements (to make sure that neither A nor B is recursive) and negative requirements (to preserve the equalities).

The positive requirements are $\{[e] \neq A \mid e \in \omega\} \cup \{[e] \neq B \mid e \in \omega\}$ and, after being interlaced, are denoted by $\{R_i \mid i \in \omega\}$.

The negative requirements are $\{[(e)_0]^A = [(e)_1]^B \mid e \in \omega\}$ and are denoted by $\{Q_i \mid i \in \omega\}$.

If R_i is $[e] \neq A$, then it is associated with $3e+1$

If R_i is $[e] \neq B$, then it is associated with $3e+2$

If Q_i is $[(e)_0]^A = [(e)_1]^B$, then it is associated with $3e$

Let $p_0(i) = \begin{cases} 3e+1 & \text{if } R_i \text{ is } [e] \neq A \\ 3e+2 & \text{if } R_i \text{ is } [e] \neq B \end{cases}$

$p_1(i) = 3e \quad \text{if } Q_i \text{ is } [(e)_0]^A = [(e)_1]^B$

Let α, β be two requirements, α associated with $p(\alpha)$ and β associated with $p(\beta)$. We say that α has higher priority than β iff $p(\alpha) < p(\beta)$.

Followers are appointed for the sake of R_i at certain stages; they are subject to cancellation at later stages. At every stage a follower is either realized or unrealized and each R_i has at most one unrealized follower.

p follows R_e iff p is appointed to follow R_e and is never cancelled. p follows R_e at stage σ $(\sigma < \alpha)$ if p was appointed prior to stage σ and was not cancelled prior to stage σ: a cancelled follower may never be reappointed! p has higher rank than q (at stage σ) if p follows R_i (at stage σ), q follows R_j (at stage σ), and R_i has higher priority than R_j (i.e. $p_0 i < p_0 j$)

p has higher order than q (at stage σ) if p and q both follow R_i (at stage σ) and p was appointed before q.

Definition 2.7

Assume R_i is $[e] \neq A$. p satisfies R_i at stage σ if p follows R_i at stage σ, $[e]_\sigma(p)$ is defined, $[e]_\sigma(p) \neq A^\sigma(p)$ and

either $A^\sigma(p) = 1$ and p was realized at stage σ

or $A^\sigma(p) = 0$ and $p \notin \{A^\gamma \mid \gamma < \sigma\}$

(A^σ is the set of ordinals put into A prior to stage σ and is identified with the characteristic function of this set; $A^\sigma(p) = 1$ means $p \notin A^\sigma$, $A^\sigma(p) = 0$ means $p \in A^\sigma$).

R_e is satisfied at stage σ if there is a p such that p satisfies R_e at stage σ

R_e is satisfied (before stage σ) if there is a τ ($\tau < \sigma$) such that R_e is satisfied at stage τ.

Definition 2.8

We need now two auxiliary functions, L and M defined as follows:

$$L(\sigma,e) \begin{cases} = \text{least } x \leq \sigma \text{ such that either } [(e)_0]_\sigma^{A^\sigma}(x)\uparrow \\ \quad \text{or } [(e)_1]_\sigma^{B^\sigma}(x)\uparrow \text{ or } (\exists p,q)([(e)_0]_\sigma^{A^\sigma}(x) = q \\ \quad \& \ [(e)_1]_\sigma^{B^\sigma}(x) = p \ \& \ q \neq p) \\ \quad \text{if there is such an } x \\ = \sigma \text{ otherwise} \end{cases}$$

$$M(\sigma,e) = \sup_{\tau \leq \sigma} L(\tau,e)$$

Definition 2.9

A follower p is associated with Q_i (at stage σ) if there is a stage τ ($\tau < \sigma$) such that p is associated with Q_i at stage τ of the construction and the association is not cancelled at any subsequent stage (and prior to σ).

Assume Q_i is $[(i)_0]^A = [(i)_1]^B$. σ satisfies Q_i iff $L(\sigma,i) = M(\sigma,i)$.

Definition 2.10

Let R_e be $[i] \neq A$ or $[i] \neq B$.

R_e <u>requires attention</u> through p at stage σ iff p follows R_e at stage σ, R_e is not satisfied prior to stage σ, $e \leq \sigma$ and at least one of the next three clauses holds:

1) p is a realized follower of R_e at stage σ and p is not associated with any Q_j at stage σ.

2) p is a realized follower of R_e at stage σ and p is associated with some Q_j and σ satisfies Q_j.

3) p is an unrealized follower of R_e at stage σ and $[i]_\sigma(p)$ is defined.

R_e requires attention at stage σ if for some p, R_e requires attention through p at stage σ; or if $e \leq \sigma$, R_e is <u>not</u> satisfied prior to stage σ, and R_e has no unrealized follower at stage σ.

<u>Construction</u>

The construction of A and B is by stages. We need α stages.

<u>Stage 0</u>

Let $A^0 = B^0 = \emptyset$

Stage $\sigma > 0$

Case 1

No R_e requires attention at stage σ

Do nothing: $A^\sigma = \bigcup\{A^\delta \mid \delta < \sigma\}$

$$B^\sigma = \bigcup\{B^\delta \mid \delta < \sigma\}$$

Go to stage $\sigma+1$

Case 2

Let R_e be the positive requirement of highest priority which requires attention at stage σ. Let $PV = \{x \mid R_x$ has lower priority than $R_e\}$ Cancel all followers of R_x for all $x \in PV$, and all association of such followers with negative requirements. R_e is said to receive attention at stage σ

If there is no follower p such that R_e requires attention through p, then consider subcase 4; if there is such a follower, then let p be the follower of R_e of highest order at stage σ such that R_e requires attention through p as defined earlier in definition 2.10, case 1, 2 and 3.

Assume such a p exists, cancel all followers of R_e of lower order than p at stage σ and all associations of such followers with negative requirements. R_e is said to <u>receive attention through p at stage</u> σ. Adopt subcase 1, 2 or 3 respectively if R_e requires attention through p at stage σ and case 1, 2 or 3 of definition 2.10 holds.

<u>Subcase 1</u>

Let $T_e = \{<y,n> \mid y < p_0(e) \ \& \ n < \omega\}$

Wellorder T_e by $<y,n> \leq <u,m>$ iff $n < m$ or $(n = m \ \& \ y \leq u)$

Let $V_e^\sigma(p) = \{<y,n> \mid <y,n> \in T_e$ & for some z, τ, u and m, $\tau < \sigma$ and p is associated with Q_z at stage τ through $<u,m>$ and $<y,n> \leq <u,m>\}$ (The association of a follower with a negative requirement will always go through some $<u,m>$ as specified below).

Let $<y_0,n_0>$ be the least element of $T_e - V_e^\sigma(p)$ such that $(\exists z \in \omega)[z < \sigma \ \& \ p_1(z) = y_0$ & no q, follower of R_e of higher order than p is associated with Q_{z_0}, where $z_0 = p_1^{-1}(y_0) = \frac{y_0}{3}]$

<u>If</u> $<y_0,n_0>$ is well defined, <u>then</u> associate p with Q_{z_0} through $<y_0,n_0>$.

Let $A^{\sigma} = \{A^{\delta} \mid \delta < \sigma\}$ and $B^{\sigma} = \{B^{\delta} \mid \delta < \sigma\}$

Go to $\sigma+1$

<u>If</u> $\langle y_0, n_0 \rangle$ is not well defined, <u>then</u>

if R_e is $[i] \neq A$, let $A^{\sigma} = \{A^{\delta} \mid \delta < \sigma\} \cup \{p\}$

and $B^{\sigma} = \{B^{\sigma} \mid \delta < \sigma\}$

if R_e is $[i] \neq B$, let $A^{\sigma} = \{A^{\delta} \mid \delta < \sigma\}$

and $B^{\sigma} = \{B^{\sigma} \mid \delta < \sigma\} \cup \{p\}$

Cancel all followers of R_e at stage σ, save for p and all associations of such followers with negative requirements.

Go to $\sigma+1$

<u>Subcase 2</u>

Assume p is associated with Q_j at stage σ. Cancel the association of p with Q_j and proceed as in subcase 1.

<u>Subcase 3</u>

p is now realized.

If $[i](p) \neq 1$, add nothing to A or B and cancel all followers of R_e at stage σ, save for p, and all associations of such followers with negative requirements, and go to $\sigma+1$.

If $[i](p) = 1$, proceed as in subcase 1.

Subcase 4

Define p to be σ. R_e receives attention through p at stage σ. Make p an unrealized follower of R_e. Add nothing to A or B.

Go to $\sigma+1$.

End of the construction.

Definition 2.11

R_e is discharged (at stage σ) if R_e does not receive attention at stage τ, for any $\tau \geq \sigma$.

R_e is discharged by p (at stage σ) if R_e does not receive attention through p at stage τ, for any $\tau \geq \sigma$

Lemma 2.9

Each R_e is discharged.

Proof

Assume all R_i's, $i < e$, have been discharged prior to stage σ.

Suppose that R_e is not satisfied at any stage.

Each realized follower of R_e at stage $\tau \geq \sigma$

is associated with a different Q_u at the end of stage $\tau \geq \sigma$. Consequently R_e has at most $p_0(e)$ realized followers (at the end of stage $\tau \geq \sigma$) and at most one unrealized follower at stage τ.

Let q_0 be the first follower of R_e of order 0 at any stage after σ. Then q_0 is never cancelled and R_e is never satisfied. If q_0 is always unrealized, then R_e is discharged, otherwise q_0 is associated with some $Q_w(3w < p_0(e))$ for all sufficiently large stages. Let σ_0 be the stage at which q_0 is last associated with Q_w. At stage σ_0+1 a follower q_1 of R_e of order 1 is appointed, never to be cancelled. And so on until termination with at most $q_{p_0(e)+1}$. Either R_e is satisfied or some q_i is never realized $(i \leq p_0(e) + 1)$. In either event R_e is discharged. ⊣

Lemma 2.10

Neither A nor B is recursive.

Proof

Assume that $A = [e]$ for some e

If p is an unrealized follower of $[e] \neq A$, at stage σ, then p is either cancelled or eventually realized (If p remains eternally unrealized, then

[e](p) would diverge while A(p) = 0 or 1 according to $p \in A$ or $p \notin A$, $\therefore$ $A \neq$ [e]⇏⇍).
By lemma 2.9 "[e] $\neq$ A" is discharged.

$\therefore$ There is a stage σ such that "[e] $\neq$ A" does not require attention at stage τ, for any $\tau \geq \sigma$.

"[e] $\neq$ A" does not have an unrealized follower at stage σ (otherwise it would not be satisfied at stage σ, hence never satisfied and either p would be cancelled or $[e]_\tau(p)$ would be defined for some $\tau > \sigma$ and so "[e] $\neq$ A" would require attention after stage σ). Since "[e] $\neq$ A" has no unrealized follower at stage σ and does not require attention at stage σ, "[e] $\neq$ A" must be satisfied prior to stage σ.

$\therefore$ There is a p that follows "[e] $\neq$ A" such that $p \in A \Leftrightarrow [e](p) = 1$

$\therefore$ $[e](p) \neq A(p)$ ⇏⇍ ⊣

Lemma 2.11

Let p follow R_i at stage σ

Let q follow R_j at stage σ

($\therefore$ p and q are both in existence at stage σ)

Assume that $p,q \in (A \cup B) - (A^\sigma \cup B^\sigma)$

Let σ_1 be the first stage at which p enters $A \cup B$, and σ_2 be the first stage at which q enters $A \cup B$.

<u>If</u> $\sigma_1 < \sigma_2$

<u>Then</u> $p_0(i) > p_0(j)$

<u>Proof</u>

As q is put in A B at stage $\sigma_2 > \sigma_1$, q is still in existence (i.e. appointed and not cancelled) at stage σ_1, since a cancelled follower can never be reappointed, and q is not cancelled at stage σ_1,

$\therefore$ R_j must have higher priority than R_i

This can be reformulated as follows:

<u>If</u> $p \in A \cup B$ and p is put in $A \cup B$ <u>before</u> q is put in $A \cup B$ or is cancelled, and p, q are both followers at some "early" stage.

($\therefore$ p enters $A \cup B$ at a stage σ such that q is still in existence at the end of stage σ but not yet in $A \cup B$)

<u>Then</u> p has been appointed for the sake of a positive requirement of lower priority than the positive requirement for which q has been appointed.

Definition 2.12 .

A set $A \subseteq \alpha$ is said to be F-regular iff for all $\sigma < \alpha$, $A \cap \sigma$ is F-subfinite (namely $A \cap \sigma$ is given by a set of subindividuals which is recursive in F and a notation for σ).

Lemma 2.12

A and B are F-regular.

Proof (for A, similar for B)

Fox $\sigma < \alpha$. To see that $A \cap \sigma$ has the required properties, it is enough to show that there are only finitely many ordinals $z < \sigma$ which can enter A at some stage $\tau > \sigma$. But if $z \in A \cap \sigma$, then $z \in A^{\sigma}$ or z is a follower at stage σ.

Let $\sigma_0 \geq \sigma$ be the least stage such that some $z < \sigma$ is placed in A at stage σ_0. For each $i \in \omega$, let σ_{i+1} be the least stage $\geq \sigma_i$, such that some $z < \sigma$ is placed in A at stage σ_{i+1}. Suppose that σ_i is well defined for all $i \in \omega$. Let R_{k_i} be the requirement satisfied at stage σ_i. As all these z's where existing followers at stage σ, by lemma 2.11 we get:

$p_0(k_0) > p_0(k_1) > p_0(k_2) > \ldots$ $\not\Leftarrow\!\!\Rightarrow$

If σ_0 is not defined, then $z \in A \cap \sigma \Leftrightarrow z \in A^{\sigma} \cap \sigma$

If σ_n is the last welldefined σ_i (and by the previous observation there is such a σ_n), then

$z \in A \cap \sigma \Leftrightarrow z \in A^{\sigma_n} \cap \sigma$ ⊣

It is now obvious that $[e]^A$ and $[e]^B$ will be single valued functions, for all $e \in \omega$.

Lemma 2.13

If $C = [(i)_0]^A = [(i)_1]^B$

Then C is recursive.

Proof

Let σ_1 be the least σ such that R_y has been discharged prior to stage σ, for every y such that $p_0 y < p_1 i = 3i$.

The existence of σ_1 follows from the proof of lemma 2.9.

Any requirement that receives attention at stage $\tau \geq \sigma_1$ has its followers subject to association with Q_i at stage τ.

To decide whether or not $x \in C$, search for a stage $\sigma_2 \geq \sigma_1$ such that $L(\sigma_2, i) = M(\sigma_2, i) > x$. σ_2 exists by lemma 2.12.

$$\therefore \quad [(i)_0]_{\sigma_2}^{A^{\sigma_2}}(x) = [(i)_1]_{\sigma_2}^{B^{\sigma_2}}(x) = q \quad \text{for some } q.$$

By regularity we have:

$$[(i)_0]^A(x) = \lim_{\sigma \to \alpha} [(i)_0]_\sigma^{A^\sigma}(x) \text{ and } [(i)_1]^B(x) = \lim_{\sigma \to \alpha} [(i)_1]_\sigma^{B^\sigma}(x)$$

$\therefore$ To show that $C(x) = q$, it is enough to show that $([(i)_0]_\tau^{A^\tau}(x) = q$ or $[(i)_1]_\tau^{B^\tau}(x) = q)$ for all $\tau \geq \sigma_2$.

Part I

Let c_0 be the computation of $[(i)_0]_{\sigma_2}^{A^{\sigma_2}}(x)$ and d_0 be the computation of $[(i)_1]_{\sigma_2}^{B^{\sigma_2}}(x)$.

c_0 (d_0 respectively) will be invalid at stage $\tau > \sigma_2$ only if some $z < \sigma_2$ is put in A (B respectively) before τ but after σ_2. Let τ_1 be the least $\tau \geq \sigma_2$ such that some $z < \sigma_2$ is put in A or B at stage τ. Let z_1 be a z put in A at stage τ_1, let z_1 follow R_{y_1} ($\therefore p_0(y_1) = 3y_1 + 1$).

Then, the computation d_0 is still valid at stage $\tau_1 + 1$.

<u>Claim</u>

<u>If</u> p is a follower of R_v such that

1) p is in existence at the end of stage τ_1

2) R_v is not satisfied before the end of stage τ_1

<u>Then</u> $p < \sigma$

<u>Proof of the claim</u>

Let R_v be a requirement such that R_v has a follower p at the end of stage τ_1 and is not satisfied before the end of stage τ_1.

$\therefore$ $p_0(v) < p_0(y_1)$ * (by lemma 2.11)

Assume now that $p \geq \sigma_2$.

$\therefore$ there is a δ such that $\sigma_2 \leq \delta < \tau_1$ and p is appointed to follow R_v at stage δ. Since $z_1 < \sigma_2$, z_1 has been appointed before σ_2; as z_1 will enter A at stage τ_1, z_1 is not cancelled at stage δ.

$\therefore$ $p_0(v) > p_0(y_1)$ because z_1 is not cancelled because of the appearing p. But this contradicts * .

$\therefore$ all followers in existence at the end of stage τ_1, for the sake of requirements which are not satisfied <u>before</u> the end of stage τ_1, must be smaller than σ_2

claim ⊣

Part II

<u>If</u> no $z < \sigma_2$ is put in B after τ_1, <u>then</u> the computation d_0 is valid forever and: (for all $\tau > \tau_1$) $[(i)_1]_\tau^{B^\tau}(x) = q$. So assume that there is a $\tau > \tau_1$ such that some $z < \sigma_2$ is put in B^τ. Let the least such τ be τ_2, and let z_2 be such a z. Suppose z_2 follows R_{y_2}. Tavo cases occur.

Case 1

z_2 is not associated with Q_i at any stage τ such that $\tau_1 \leq \tau \leq \tau_2$. Since $z_2 < \sigma_2 < \tau_1$, z_2 existed as a follower at the end of stage τ_1. If z_2 was unrealized at the end of stage τ_1, then z_2 was eligible for association with Q_i through $\langle 3i,0\rangle$; and if z_2 was associated with some Q_j through $\langle w,n\rangle$ at the end of stage τ_1, then z_2 was eligible for association with Q_i through $\langle 3i,n+1\rangle$. But since z_2 is <u>not</u> associated with Q_i we have:

some follower r of R_{y_2} of higher order than that of z_2 was associated with Q_i at stage ρ, where ρ is the least stage such that z_2 was associated with some Q_j through some $\langle u,k\rangle$ (if there is no such ρ, let $\rho = \tau_2$).

We have $\langle u,k\rangle > \langle 3i,0\rangle$ if z_2 was unrealized at stage τ_1, and $\langle u,k\rangle > \langle 3i,n+1\rangle$ if z_2 was realized at stage τ_1.

But since r has higher order than z_2 at stage ρ, $r < z_2$.

$\therefore$ r was in existence at stage σ_2.

If r were not associated with Q_i at stage σ_2, then R_{y_2} must have received attention through r at some stage δ $(\sigma_2 \leq \delta < \rho)$ [because r was not associated with Q_i at stage σ_2, but will be at stage $\rho \leq \tau_2$].

But z_2 would have been cancelled at stage δ and hence could not be a follower at stage τ_2.

$\therefore$ r is associated with Q_i at stage σ_2.

But σ_2 is such that $L(\sigma_2,i) = M(\sigma_2,i) > x$, i.e. Q_i is satisfied by σ_2.

$\therefore$ r is a follower of R_{y_2} which is associated to a negative requirement at stage σ_2.

$\therefore$ R_{y_2} requires attention through r at stage σ_2.

There are only 3 possibilities.

1) R_{y_2} receive attention through r

$\therefore$ z_2 is cancelled $\Rightarrow\Leftarrow$

2) R_{y_2} receive attention through a follower of higher higher order than r

$\therefore$ z_2 is cancelled $\Rightarrow\Leftarrow$

3) R_v requires attention at stage σ_2 with $p_0(v) < p_0(y_2)$

$\therefore$ z_2 is cancelled $\Rightarrow\Leftarrow$

Hence case 1 cannot occur.

<u>Case 2</u>

z_2 is associated with Q_i at stage τ for some τ such that $\tau_1 \leq \tau \leq \tau_2$; let τ_1' be the least such τ.

Since z_2 enters B at stage τ_2, there is a first stage τ_1'', such that $\tau_1' < \tau_1'' \leq \tau_2$ and the association of z_2 with Q_i is cancelled at stage τ_1''.

Since z_2 is still a follower at the end of stage τ_1'', $L(\tau_1'',i) = M(\tau_1'',i)$ [otherwise Q_i is not satisfied by τ_1'' $\therefore$ R_{y_2} does not receive attention through z_2 at stage τ_1'' $\therefore$ we cannot cancel the association of z_2 with Q_i]

and there is a computation c_1 of $[(i)_0]^{A_{\tau_1''}}_{\tau_1''}(x) = q_1$

(assume $[(i)_0]^{A_{\tau_1''}}_{\tau_1''}(x) \uparrow$, as $x < L(\sigma_2,i) \leq \sigma_2 < \tau_1''$

we have: $M(\tau_1'',i) = L(\tau_1'',i) \leq x < L(\sigma_2,i) \leq M(\tau'',i)$

which is a contradiction. Hence:

$$L(\tau_1'',i) = M(\tau_1'',i) \Rightarrow [(i)_0]^{A_{\tau_1''}}_{\tau_1''}(x) \downarrow)$$

<u>If</u> c_1 were invalid at the end of stage τ_2, <u>then</u> some follower x would land in A at some stage λ $(\tau_1'' < \lambda < \tau_2)$

Suppose x follows R_v. $x < \tau_1''$ and x exists as a follower at stage τ_1''. Also $p_0(v) < p_0(y_2)$ because R_{y_2} has a follower z_2 which exists at stage τ_1'', R_{y_2} requires attention through z_2 but does not cancel x (which enters A <u>after</u> stage τ_1'' and has been appointed <u>before</u> stage τ_1''). x has to be $< \tau_1''$, otherwise it would not invalidate c_1. But as R_v receives attention at stage λ, z_2 would be cancelled at stage $\lambda < \tau_2$. $\Rightarrow\Leftarrow$

Hence c_1 is valid at the end of stage τ_2.

But all followers in existence at the end of stage τ_2 are less than σ_2 (By the same argument as in part I, with z_2 for z_1, τ_2 for τ_1, B for A)

Let $c_2 = c_1$

If no $z < \sigma_2$ is placed in A after stage τ_2, then the computation c_2 is valid forever. Assume that some $z < \sigma_2$ is placed in A after stage τ_2.

Continue to alternate as above between A and B.

If for some $n \in \omega$, z_n fails to be defined, then the lemma (and thus theorem 2.8) is proved. Suppose z_n is defined for all $n \in \omega$. z_n follows R_{y_n} at stage σ_2. By lemma 2.11 we have:

$$p_0(y_1) > p_0(y_2) > \dots \qquad \Rightarrow\Leftarrow$$

3. A minimal pair of type n+2 objects.

As in section 2 we shall assume that [illegible] is a fixed integer and that F is a type n+2 object of the form $\langle F', {}^{n+2}E \rangle$ where F' is some fixed type n+2 object. As in the previous section we denote by α the ordinal λ^F_{n-1}.

In the previous chapter, we defined reducibilities in such a way that the construction of a minimal pair of subsets of α was easy.

We want now to get two type n+2 objects A and B such that none of them is recursive in the other and such that, for every type n+1 object D corresponding to a set of ordinals, the following is true: if D is recursive in both A and B, then D is recursive. To reach this result we shall define new reducibilities which in turn will make our work "easy". This will be done in Harrington's universe.

Harrington has defined reducibilities [8] which he used in his thesis to solve, in a Shore-like way (see [27]), Post's problem. Unluckily such reducibilities require too many requirements. The success of the construction would then depend upon the fact that α is (or is not) refractory.

We do not know whether α is always non-refractory. It is easy to see that under $V = L$, α is projectible into $\aleph_{n-1} < \alpha$ and α is thus not refractory. A similar result can be obtained under AD (for a suitable value of n) as shown by Moschovakis in [19], but as we assume that AC holds throughout this thesis, (and as we do not believe in AD) we reject any attempt to use AD.

Nevertheless, Harrington's idea can be used to define a reducibility (not as fine as his) such that we shall have only to take care (as in chapter 2) of countably many requirements.

As in the previous chapter we will be able to list these requirements "in advance", i.e. the priority of each requirement will be given by a recursive function and there will be no problem in proving this version of the minimal pair. The type n+2 objects that we shall get, will be countable collection of H_σ^F sets (where the H_σ^F are the set used to define h-recursion). The objects that we shall build will be such that every computation from them will stop before κ_{n-1}^F, thus by corollary 1.3 (and definition 1.2) it will be enough to solve the problem for subsets of α.

We must first give a few definitions

Definition 3.1

For $a \in I$ and $\sigma < \kappa^F$, we call σ an a-reflecting ordinal iff for all Σ_1 formulas, $\phi(X)$, of L we have

$M_\sigma(F) \models \phi(a) \Rightarrow M_{\kappa_0^a}(F) \models \phi(a)$

It is easy to see that κ_{n-1}^F is a-reflecting (Grilliot) but that κ^F is not (Moschovakis)

Definition 3.2

An ordinal σ will be called recursive in F

iff H_σ^F is recursive in F

iff there is a prewellordering of I of length σ, recursive in F

An ordinal σ will be called constructive in F

iff σ has a notation from an integer

iff $(\exists m \in \omega)(\langle m,0\rangle \in \mathcal{O}^F \ \& \ |\langle m,0\rangle|^F = \sigma)$

An ordinal σ will be called subrecursive (subconstructive) in F if for some $a \in SI$, σ is recursive (constructive) in $\langle F,a\rangle$

If the type n+2 functional F is clearly understood, then for $a \in I$, an ordinal σ will be called subrecursive (subconstructive) in a iff σ is subrecursive (subconstructive) in $\langle F,a\rangle$.

For $a \in I$, let $P^a_{n-1} = \{i \in SI \mid i \in \mathcal{O}^{\langle F,a\rangle}\}$

P^a_{n-1} is a complete semirecursive in F subset of SI. It is clear that <u>given $\langle F,a\rangle$</u> P^a_{n-1} and $H^F_{\kappa^a_{n-1}}$ are recursively equivalent and hence that both epitomize the structure $M_{\kappa^a_{n-1}}(F)$.

<u>Theorem 3.1</u> (<u>Harrington</u>)

Let $a \in I$ and $G = \langle P^a_{n-1}, a\rangle$

Let $\sigma = \kappa^G_{n-1}$

Then σ is a-reflecting ⊣

This can be restated as follows:

<u>Corollary 3.2</u> (<u>Kechris</u>)

Let $a \in I$. <u>If</u> $\mathcal{B}$ is a collection, semirecursive in $\langle F,a\rangle$, of subsets of SI and <u>if</u> $\mathcal{B}$ contains at least one subset of SI which is semirecursive in $\langle F,a\rangle$, <u>then</u> $\mathcal{B}$ contains a subset of SI which is recursive in $\langle F,a\rangle$ ⊣

Note that the limit of a-reflecting ordinals is also a-reflecting. Let κ_r^a be the last a-reflecting ordinal. Let $\kappa_r = \kappa_r^0$.

It is clear that metarecursion can be defined in terms of recursion in ${}^2E$: to be more precise a subset A of ω_1^{CK} is meta-r.e. iff the type 2 object $\{H_\sigma^{{}^2E} \mid \sigma \in A\}$ is semirecursive in ${}^2E$. Harrington's idea was to lift this definition to an arbitrary (normal) type n+2 object (with, of course, $n \geq 1$).

Definition 3.3

Let S be the set of ordinals subconstructive in F

So $S = \{|j|^F \mid j \in SI \cap \mathcal{O}^F\}$

For $A \subseteq S$, let $\overline{A} = \{H_\sigma^F \mid \sigma \in A\}$.

$\overline{A} \subseteq Tp(n+1)$ and thus can be viewed as an object of type n+2.

Note that $\overline{S}$ is recursive in F

Definition 3.4

Given $A, B \subseteq S$, A will be called F-r.e. iff $\overline{A}$ is semirecursive in F. A is said to be F-calculable in B iff $\overline{A}$ is recursive in $\langle F,\overline{B}\rangle$.

This is our main divergence from Harrington's definition: he required only $\overline{A}$ to be recursive in $\langle F,\overline{B},H^F_\sigma\rangle$ for some $\sigma \in S$.

We will show that the main lemmas remain true (modulo certain new definitions) and that this gives us the possibility to use at most countably many requirements. We will try now to reduce this "higher type" problem to the setting of "α-recursion" theory.

Definition 3.5

As in chapter 2 we have:

$\alpha = \lambda^F_{n-1}$ = order type of S

t is the unique order preserving isomorphism between α and S

$\therefore$ for any $\beta < \alpha$, $t(\beta) \in S$ and $t(\beta) < \kappa^F_{n-1}$

$t(\beta)$ is the β^{th} ordinal having a notation from a subindividual

Define $T \subseteq \omega \times \alpha \times \alpha$ as follows:

$T = \{\langle e,\beta,\sigma\rangle \mid$ e is the Gödel number of a Σ_1 formula $\phi(X)$ in L, $\beta < \sigma$ and $M_{t(\sigma)}(F) \models \phi(t(\beta))\}$

Let $\mathfrak{A}$ be the structure $\langle L_\alpha(T),\varepsilon,T\rangle$

Definition 3.6 (Reducibilities)

Assume $A \subseteq S$, $\beta_0 < \alpha$, $\beta_1 < \alpha$ and $e \in \omega$

Let $G = \langle F, \bar{A} \rangle$

Then $[e]^A(\beta_0) \simeq \beta_1$ iff $(\exists p, x \in S)(\{e\}^G(p) \simeq x$

$\& \ |p|^F = t\beta_0 \ \& \ |x|^F = t\beta_1)$

iff there is an ordinal $\sigma < \kappa_0^G$ such that $M_\sigma(G) \models \phi_{e_1}(t\beta_0, t\beta_1)$ where ϕ_{e_1} is the Σ_1 formula describing the graph of the $e^{\underline{th}}$ recursive function

Remark $\sigma < \kappa_0^G$ but it is not necessarily the case that $\sigma < \kappa_{n-1}^F$

Let $[e]_\tau^A(\beta_0) \simeq \beta_1$ iff $(\exists p, x \in S)(\{e\}^G(p) \simeq x$ $\& \ |\langle e_0, p\rangle|^G < t(\tau)$ iff $(\exists \rho < t(\tau))[M_\rho(G) \models \phi_{e_1}(t\beta_0, t\beta_1)]$

Before we go further, we shall need a few facts.

Lemma 3.3 (Harrington)

$A \subseteq \alpha$ is Σ_1 over $\mathfrak{A}$ iff $t[A]$ is F-r.e.

Proof

<= If $t[A] \subseteq S$ is F-r.e.

Then there is a Σ_1 formula $\phi(X)$ of L with Gödel number e such that for $\sigma < \alpha$, we have:

$t\sigma \in t[A]$ iff $M_{\kappa^F_{n-1}}(F) \models \phi(t\sigma)$

iff $(\exists\gamma \in S)(\gamma > t\sigma \; \& \; M_\gamma(F) \models \phi(t\sigma))$

iff $\langle e,\sigma,t^{-1}\gamma\rangle \in T$

$\Rightarrow$ Conversely given $\sigma \in S$, $S \cap \sigma$ is first order definable over $M_\sigma(F)$ uniformly, and thus, letting $\delta = t^{-1}\sigma$, the structure $\langle L_\delta(T),\varepsilon,T \cap (\omega \times \delta \times \delta)\rangle$ is first order definable over $M_\sigma(F)$ uniformly. Thus a Σ_1 definition over $\mathfrak{A}$ induces one over $M_{\kappa^F_{n-1}}(F)$ (Remember: $\kappa^F_{n-1} = \sup S$)

$\dashv$

Clearly this implies that $\mathfrak{A}$ is admissible.

This lemma will allow us to use the recursion theory which $\mathfrak{A}$ induces on α, and then to apply the α-finite injury method.

Definition 3.7

For $A,B \subseteq \alpha$, let $A \leq_{\mathfrak{A}} B$ mean: "A is $\underset{\sim}{\Delta}_1$ definable over the structure $\langle\mathfrak{A},B\rangle$"

We are now ready to state the main result of this section:

Theorem 3.4

There are two semirecursive in F sets of type n+1 objects such that neither is recursive in the other and such that every object, corresponding to a set of ordinals less than α, of type n+1, recursive in both of them is recursive in F.

Using the reducibilities that we defined a few lines ago, we will get two sets $t^{-1}A$, $t^{-1}B \subseteq \alpha$, both Σ_1 definable over $\mathfrak{A}$ and such that neither of A, B is F-calculable in the other and such that every subset $t^{-1}C$ of α, with C F-calculable in both A and B, is such that C is F-calculable in $\emptyset$.

The only problem that could arise while we use the ordinals to solve the problem would be the following: "the ordinal σ that we need to be able to see that $[e]^A_\sigma(\beta_0) \simeq \beta_1$, for instance, might be bigger than κ^F_{n-1}." In this case, we would not get any interesting information, inside $M_{\kappa^F_{n-1}}(F)$, about objects of type n+1. Hence we must try to find some other properties which, if possessed by A, B $\subseteq \alpha$, imply that the computations do not become "too wild" or "too long".

Definition 3.8

A subset, A, of α is $\mathfrak{A}$-(hyperregular and regular) if the structure $\langle \mathfrak{A}, A \rangle$ is admissible.

A subset B of S if F-subgeneric iff $\kappa_0^{\langle\overline{B},F\rangle} \leq \kappa_{n-1}^F$.

(This is also a place where our definition is noticeably different from Harrington's: he requires that for all $\sigma \in S$ $\kappa_0^{G(\sigma)} \leq \kappa_{n-1}^F$ where $G(\sigma) = \langle\overline{B},F,H_\sigma^F\rangle$, we only require it for $\sigma = 0$)

Lemma 3.5

Given $B \subseteq \alpha$, if B is $\mathfrak{A}$-(hyperregular and regular) and if t[B] is F-subgeneric, then for all $A \subseteq \alpha$ we have:

$A \leq_{\mathfrak{A}} B$ iff t[A] is f-calculable in t[B]

Proof:

=> This too is fairly easy to see: F-calculable in t[B] is always at least as strong as ... $\leq_{\mathfrak{A}} B$.

<= If t[A] is F-calculable in t[B], then for any $\sigma \in S$, the relations $\sigma \in t[A]$ and $\sigma \notin t[A]$ correspond to Σ_1 (with some member(s) of S as parameter) assertions about the structure $M_\beta(G)$ where $\beta = \kappa_0^G$ and $G = \langle\overline{t[B]},F\rangle$. Thus, since t[B] is F-subgeneric we have that t[A] is Δ_1 (with some member(s) of S

as parameter) over $M_{\kappa_{n-1}^F}(G)$. So we just need to show that the fact that B is $\mathfrak{A}$-(hyperregular and regular) implies that this induces a $\underset{\sim}{\Delta}_1$ definition of A over $<\mathfrak{A},B>$.

By the proof of lemma 3.3, this will be the case if $\{<e,\beta,\sigma> \mid \beta < \sigma < \alpha$ & e is the Gödel number of a Σ_1 formula $\phi(x)$ in L & $M_{t\sigma}(<\overline{t[B]},F>) \quad \phi(t\beta)\}$ is Δ_1 over $<\mathfrak{A},B>$. So given $\sigma < \alpha$, since $<\mathfrak{A},B>$ is admissible, $\sigma \cap B \in L_\alpha(T)$. Thus for some $\delta < \alpha$ $(\delta > \sigma)$, $t[B] \cap t\sigma$ is first order definable over $M_{t\delta}(F)$, say by a formula with Gödel number e_0. Hence $t[B] \cap t\sigma \in M_{\kappa_{n-1}^F}(F)$, and thus: $M_{t\sigma}(G) \in M_{\kappa_{n-1}^F}(F)$ with $G = <\overline{t[B]},F>$

So given $\beta < \alpha$ and given a formula $\phi(x)$ in L, $M_{t\sigma}(G) \models \phi(t\beta)$ iff $(\exists\delta \in S)(\exists e_0 \in \omega)(\exists X,Y \in M_{\kappa_{n-1}^F}(F))$ [X is first order definable over $M_\delta(F)$ by the formula with Gödel number e_0 & $X = t[B] \cap t\sigma$ & $Y = M_{t\sigma}(<\overline{X},F>)$ & $Y \models \phi(t\beta)$]. Using the T predicate of definition 3.5, this corresponds to a Σ_1-formula over $<\mathfrak{A},B>$.

—|

Remark 3.6

<u>If</u> $B \subseteq \alpha$ is Σ_1 over $\mathcal{O}\!\mathcal{L}$, <u>then</u> if $t[B]$ is F-subgeneric, B is $\mathcal{O}\!\mathcal{L}$-(hyperregular and regular).

Hence it is enough to find some property which implies that $t[A]$ and $t[B]$ will be F-subgeneric (we assume that we build them in a Σ_1 way) <u>and</u> which is suitable for inclusion in a priority argument.

We must also get a property which does not force us to use more than countably many requirements (so we will be able to get two type n+2 objects which are a minimal pair for type n+1 objects <u>for all $n \geq 1$</u>).

Let B be Σ_1 over $\mathcal{O}\!\mathcal{L}$, think of B as being enumerated in α many steps. Let B^σ be the part of B enumerated prior to stage σ. Without loss of generality we can assume $B^\sigma \subseteq \sigma$. The property we want is following:

<u>Definition 3.9</u> <u>Property H</u>

For all Σ_1 formulas $\phi(X)$ in $L_{\mathcal{O}\!\mathcal{L}}$ there is a $\delta < \alpha$ such that:

<u>if</u> $(\exists\gamma < \alpha)(\gamma \geq \delta \;\&\; \langle L_\gamma(T), \varepsilon, T \restriction \gamma, B^\gamma\rangle \models \phi(0))$

<u>then</u> $(\exists\gamma < \alpha)(\gamma \geq \delta \;\&\; B^\gamma = B \cap \gamma \;\&\; \langle L_\gamma(T), \varepsilon, T \restriction \gamma, B^\gamma\rangle \models \phi(0))$

Property H means that, given some basic fact about our hierarchy (i.e. $\phi(0)$ with ϕ in Σ_1), there is a level δ such that if that fact happens to become true (because of some initial segment of B) after that level, then the same fact must be true at some level γ (after δ) where all the relevant part of B is perfectly known (i.e. some level γ such that for all $\tau < \gamma$, γ knows whether $\tau \in B$ or $\tau \notin B$ and no ordinal $\gamma' > \gamma$ will view $B \cap \gamma$ differently).

It is easy to work property H into a priority argument, as we will show later. But we must still prove that property H implies that $t[B]$ is F-subgeneric.

Lemma 3.7

Assume B is Σ_1 over

If B satisfies property H, then $t[B]$ is F-subgeneric.

Proof

Let B be Σ_1 over $\mathfrak{A}$ such that B satisfies property H. Let $C = t[B]$.

$\therefore$ C is F-r.e.

$\therefore$ there is a Σ_1 formula $\phi(x)$ in L such that for

$\beta < \kappa^F_{n-1}$ we have

$$\beta \in C \iff M_{\kappa^F_{n-1}}(F) \models \phi(\beta)$$

For $\sigma < \kappa^F$, let $C^\sigma = \{\beta < \sigma \mid M_\sigma(F) \models \phi(\beta)\}$

By the proof of lemma 3.3 $\phi(x)$ can be chosen so that for all $\sigma \leq \alpha$, $t[B^\sigma] = C^{t\sigma}$

Let $S' = \{\beta < \kappa^F \mid$ letting $\delta = \sup (S \cap \beta)$, we have $\delta + \beta = \beta\}$

$\therefore$ $\beta \in S'$ iff $\sup (S \cap \beta)$ is small in comparison with β.

By the proof of lemma 3.5, the fact that B satisfies property H can be translated as follows: for any Σ_1 formula $\phi(x)$ in L, there is a $\delta \in S$, such that <u>if</u> $(\exists\gamma \in S \cap S')(\gamma \geq \delta$ & $M_\gamma(\langle\overline{C^\gamma},F\rangle) \models \phi(0))$, <u>then</u> $(\exists\gamma \in S \cap S')(\gamma \geq \delta$ & $M_\gamma(\langle\overline{C^\gamma},F\rangle) \models \phi(0)$ & $C^\gamma = C \cap \gamma)$ and thus $M_{\kappa^F_{n-1}}(\langle\overline{C},F\rangle) \models \phi(0)$

Now we must show that C is F-subgeneric, i.e. given $m \in \omega$ such that $\langle m,0\rangle \in \mathcal{O}^G$, with $G = \langle\overline{C},F\rangle$, we must prove that $|\langle m,0\rangle|^G < \kappa^F_{n-1}$.

There is a $\underset{\sim}{\Sigma}_1$ formula $\psi(x)$ in L such that for any ordinal $\beta \in S'$ we have: $M_\beta(G) \models \phi(0)$ iff $\beta > |\langle m,0\rangle|^G$.

Since $\langle m,0\rangle \in O^G$, there is an ordinal β' such that $M_{\beta'}(G) \models \phi(0)$; we might as well choose β' such that $\beta' > \kappa^F_{n-1}$ & $\beta' \in S'$

$\therefore$ $\kappa^F_{n-1} + \beta' = \beta'$

$\therefore$ $C = C^{\beta'}$ and so $M_{\beta'}(\langle\overline{C^{\beta'}},F\rangle) \models \phi(0)$

By lemma 3.1 (reflexion) we may choose such a β' so that $\beta' < \kappa^F_r$ and thus, there are arbitrarily large $\gamma' < \kappa^F_{n-1}$ such that $\gamma' \in S'$ and $M_{\gamma'}(\langle\overline{C^{\gamma'}},F\rangle) \models \phi(0)$. For each such γ', let γ be the <u>first</u> τ member of S, such that $\tau \geq \gamma'$. Then $\gamma \in S'$ and $C^{\gamma'} = C^\gamma$

$\therefore$ there are arbitrarily large γ's members of $S \cap S'$, such that:

$$M_\gamma(\langle\overline{C^\gamma},F\rangle) \models \phi(0)$$

$\therefore$ the translation of property H implies that

$$M_{\kappa^F_{n-1}}(\langle\overline{C},F\rangle) \models \phi(0)$$

$\therefore$ $<m,0> \in O^G$ & $|<m,0>|^G < \kappa^F_{n-1}$

(with $G = <\overline{C},F>$).

We are ready now to attack the proof of our main result. This proof will follow the main lines of theorem 2.8. But here we must add to the positive and negative requirements, some special requirement to take care of the subgenericity. This will force us to be more careful in the construction: we need some kind of semi-cancellation: the prohibition!

Definition 3.10

The positive requirements are

$\{[e] \neq A \mid e \in \omega\} \cup \{[e] \neq B \mid e \in \omega\}$ and after being interlaced are denoted by $\{R_i \mid e \in \omega\}$.

The negative requirements are

$\{[(e)_0]^A = [(e)_1]^B \mid e \in \omega\}$ and are denoted by

$\{Q_i \mid i \in \omega\}$

Let $\Gamma^A_e = \{\beta < \alpha \mid <L_\beta(T), \varepsilon, T \restriction \beta, A^\beta> \models \phi_e(0)\}$

and $\Gamma^B_e = \{\beta < \alpha \mid <L_\beta(T), \varepsilon, T \restriction \beta, B^\beta> \models \phi_e(0)\}$

The special requirements are

$$\{\text{"}\Gamma_e^A \neq \emptyset \Rightarrow (\exists\gamma \in \Gamma_e^A)(A^\gamma = A \cap \gamma)\text{"} \mid e \in \omega\}$$

$$\cup \{\text{"}\Gamma_e^B \neq \emptyset \Rightarrow (\exists\gamma \in \Gamma_e^B)(B^\gamma = B \cap \gamma)\text{"} \mid e \in \omega\}$$

After being interlaced the special requirements are denoted by $\{S_i \mid i \in \omega\}$

<u>If</u> R_i is $[e] \neq A$, <u>then</u> it is associated with 5e+1

<u>If</u> R_i is $[e] \neq B$, <u>then</u> it is associated with 5e+3

<u>If</u> Q_i is $[(e)_0]^A \neq [(e)_1]^B$, <u>then</u> it is associated with 5e.

<u>If</u> S_i is special for B and e, <u>then</u> it is associated with 5e+2

<u>If</u> S_i is special for A and e, then it is associated with 5e+4

(S_i is special for X and e iff S_i is "$\Gamma_e^X \neq \emptyset \Rightarrow (\exists\gamma \in \Gamma_e^X)(X^\gamma = X \cap \gamma$" (where X is A or B))

Let
$$p_0(i) = \begin{cases} 5e+1 & \text{if } R_i \text{ is } [e] \neq A \\ 5e+3 & \text{if } R_i \text{ is } [e] \neq B \end{cases}$$

$$p_1(i) = 5e \quad \text{if } Q_i \text{ is } [(e)_0]^A = [(e)_1]^B$$

$$p_2(i) = \begin{cases} 5e+2 & \text{if } S_i \text{ is special for B and e} \\ 5e+4 & \text{if } S_i \text{ is special for A and e} \end{cases}$$

Let $\underset{\sim}{\alpha}, \underset{\sim}{\beta}$ be two requirements, α associated with k and β associated with $\ell(k, \ell \in \omega)$.

We say that $\underset{\sim}{\alpha}$ has <u>higher priority</u> than $\underset{\sim}{\beta}$ (and write $p(\underset{\sim}{\alpha}) < p(\underset{\sim}{\beta})$) iff $k < \ell$.

<u>Followers</u> are appointed for the sake of R_i at certain stages; they are subject to cancellation or prohibition at later stages.

A cancelled follower remains cancelled forever, but a prohibition can be <u>lifted</u>.

At every stage a follower is either <u>realized</u> or <u>unrealized</u> and each R_i has at most one unrealized follower

$p \in \alpha$ follows R_e iff p is appointed to follow R_e and is never cancelled; p follows R_e at stage σ $(\sigma < \alpha)$ if p was appointed prior to stage σ and was not cancelled prior to stage σ; we say also that such an ordinal is a follower "<u>still in existence</u>" at stage σ

p has higher <u>rank</u> than q (at stage σ) if p follows R_i (at stage σ), q follows R_j (at stage σ), and R_i has higher priority than R_j (i.e. $p_0 i < p_0 j$).

p has higher <u>order</u> than q (at stage σ) if p and q both follow R_i (at stage σ) and p was appointed before q.

<u>Definition 3.11</u>

As usual we denote by A^σ(B^σ respectively) that part of A(B) which has been enumerated prior to stage σ as well as its characteristic function.

Assume R_i is $[e] \neq A$.

p <u>satisfies</u> R_i at stage σ if p follows R_i at stage σ, $[e]_\sigma(p)$ is defined, $[e]_\sigma(p) \neq A^\sigma(p)$ and either $A^\sigma(p) = 1$ and p was realized at stage σ or $A^\sigma(p) = 0$ and $p \notin \bigcup\{A^\gamma \mid \gamma < \sigma\}$

R_i is <u>satisfied</u> at stage σ if there is a $p \in \alpha$, such that p satisfies R_i at stage σ.

R_i is satisfied (satisfied before stage σ respectively) if there is a stage τ ($\tau < \sigma$) such that R_i is satisfied at stage τ.

Similar definitions are made if R_i is $[e] \neq B$.

<u>Definition 3.12</u>

We need now two auxiliary functions:
$L: \alpha \times \omega \to \alpha$ and $M: \alpha \times \omega \to \alpha$ defined as follows:

$$L(\sigma,e) = \begin{cases} \text{least } x \leq \sigma \text{ such that either } [(e)_0]_\sigma^{A^\sigma}(x) \uparrow \\ \text{or } [(e)_1]_\sigma^{B^\sigma}(x) \uparrow \text{ or } (\exists p,q \in \alpha) \\ [(e)_0]_\sigma^{A^\sigma}(x) = q \;\&\; [(e)_1]_\sigma^{B^\sigma}(x) = p \;\&\; p \neq q) \\ \qquad\qquad\qquad \text{if there is such an } x \\ \sigma \qquad\qquad\qquad \text{otherwise} \end{cases}$$

$M(\sigma,e) = \sup \{L(\tau,e) \mid \tau \leq \sigma\}$

Definition 3.13

A follower $p \in \alpha$ is associated with Q_i (at stage σ) if there is a stage σ' ($\sigma' < \sigma$) such that p is associated with Q_i at stage σ' of the construction and the association is not cancelled at any later stage (and prior to stage σ).

Assume Q_i is $[(e)_0]^A = [(e)_1]^B$: σ satisfies Q_i iff $L(\sigma,e) = M(\sigma,e)$

Assume S_i is special for B and e

Let $\Gamma^\sigma_{e,B} = \{\gamma < \alpha \mid B^\gamma = B^\sigma \cap \gamma \;\&$

$$\langle L_\gamma(T),\varepsilon,T \restriction \varepsilon,B^\gamma\rangle \models \phi_e(0)\}$$

Let $\gamma_0 = \mu\gamma[\gamma \in \Gamma^\sigma_{e,B}]$

S_i is satisfied (at stage σ) iff all the ordinals $\beta < \alpha$ such that $[\beta < \gamma_0 \;\&\; \beta \notin B^{\gamma_0} \;\&\; \beta$ is a follower in existence (at stage σ) for some requirement R_j, with R_j a positive requirement for B (i.e. $p_0 j = 5e+3$ for some e) and with $p_0 j < p_2 i$] are prohibited.

Remark:

The construction will be such that an ordinal β, member of a prohibition at stage σ, for the sake of some S_i cannot enter A (or B) at some later stage, except if β follows some R_j of higher priority than S_i; but such an ordinal is not cancelled: it is still in existence.

Definition 3.14

Let R_e be $[i] \neq A$ or $[i] \neq B$. R_e requires attention through p at stage σ if

1) p follows R_e at stage σ

2) R_e is not satisfied prior to stage σ

3) $e \leq \sigma$ and at least one of the next three clauses hold:

4.1) p is a realized follower of R_e at stage σ and p is not associated with any Q_j at stage σ.

4.2) p is a realized follower of R_e at stage σ and p is associated with some Q_j and σ satisfies Q_j.

4.3) p is an unrealized follower of R_e at stage σ and $[i]_\sigma(p)$ is defined.

R_e <u>requires attention</u> at stage σ if for some p, R_e requires attention through p at stage σ; or if $e \leq \sigma$, R_e is not satisfied prior to stage σ and R_e has no unrealized follower in existence at stage σ.

Let S_i be special for B and e.

S_i <u>requires attention</u> at stage σ iff $i \leq \sigma$, S_i is not satisfied prior to stage σ and

$$\langle L_\sigma(T), \varepsilon, T \upharpoonright \sigma, B^\sigma \rangle \models \phi_e(0)$$

And similarly for S_j special for A and e.

<u>Definition 3.15</u> (<u>The Construction</u>)

The construction of A and B is by stages; we need α stages.

<u>Stage 0</u>

$$A^0 = B^0 = \emptyset$$

<u>Stage $\sigma > 0$</u>

<u>Case 1</u>

No R_e or S_e requires attention at stage σ

Do nothing:

Let $A^\sigma = \bigcup\{A^\delta \mid \delta < \sigma\}$

$B^\sigma = \bigcup\{B^\delta \mid \delta < \sigma\}$

Do not change the prohibitions

Go to step $\sigma+1$

<u>Case 2</u>

Some R_i or S_i requires attention

Look for the requirement of highest priority that requires attention.

2.1) Say S_e is this requirement, where S_e is special for B and i

Cancel all followers of all positive requirements of lower priority and all associations of such followers with negative requirements.

S_e is said to <u>receive attention</u> at stage σ

Prohibit all ordinals β (for membership in B) such that $(\beta < \gamma_0 = \mu\gamma[\gamma \in \Gamma^\sigma_{i,B}])$ & $(\beta \notin \beta^{\gamma_0})$ & (β follows (at stage σ) some positive requirement R_j (where R_j is $[u] \neq B$)) [i.e.: all $\beta < \gamma_0$ such that $\beta \notin \beta^{\gamma_0}$ and β is a follower still in existence of a positive requirement mentioning B].

A similar construction exists if S_e is special for A and i.

2.2) Say R_e is this requirement.

Cancel all followers for positive requirement R_y with $p_0y > p_0e$, and all associations of such followers with negative requirements and <u>lift</u> all prohibitions for special requirements S_z with $p_2z > p_0e$.

R_e is said to receive attention at stage σ

If there is no follower p such that R_e requires attention through p at stage σ, then consider subcase 4. If there is such a follower, then let p be the follower of R_e of highest order at stage σ such that R_e requires attention through p as defined earlier in (4.1), (4.2) and (4.3).

Assume such a p exists.

Cancel all followers of R_e of lower order than p at stage σ and all associations of such followers with negative requirements. R_e is said to receive attention through p at stage σ. Adopt subcase 1, 2 or 3 respectively if R_e requires attention through p at stage σ and clause (4.1), (4.2) or (4.3) holds, respectively.

Subcase 1

Let $T_e = \{\langle y,n\rangle \mid y < p_0 e \ \& \ n < \omega\}$

Well order T_e by $\langle y,n\rangle \leq \langle u,m\rangle$ iff $n < m$ or $n = m \ \& \ y \leq u$.

Let $V_e^{\sigma}(p) = \{\langle y,n\rangle \in T_e \mid$ for some z, σ', u and m we have $[\sigma < \sigma'$ and p is associated with Q_z at stage σ' through $\langle u,m\rangle$ and $\langle y,n\rangle \leq \langle u,m\rangle\}$.

The association of a follower with a negative requirement will always take place through some $\langle u,m\rangle$ as specified below.

Let $\langle y_0,n_0\rangle$ be the least member of the set $T_e - V_e^{\sigma}(p)$ such that $(\exists z)(z \in \omega \ \& \ z < \sigma \ \& \ p_1(z) = y_0 \ \&$ no follower of R_e of higher order than p (at stage σ) is associated with Q_z).

If $\langle y_0,n_0\rangle$ is well defined, then associate p with Q_{z_0} through $\langle y_0,n_0\rangle$ where z_0 is the unique z such that $z < \sigma$ and $p_1(z_0) = y_0$.

Let $A^{\sigma} = \bigcup\{A^{\delta} \mid \delta < \sigma\}$

$B^{\sigma} = \bigcup\{B^{\delta} \mid \sigma < \sigma\}$

Go to stage $\sigma+1$

If $\langle y_0, n_0\rangle$ is not well defined and R_e is $[i] \neq A$, let $A^\sigma = \bigcup\{A^\delta \mid \delta < \sigma\} \cup \{p\}$ and $B^\sigma = \bigcup\{B^\delta \mid \delta < \sigma\}$; if R_e is $[i] \neq B$, let $B^\sigma = \bigcup\{B^\delta \mid \delta < \sigma\} \cup \{p\}$ and $A^\sigma = \bigcup\{A^\delta \mid \delta < \sigma\}$.

Go to stage $\sigma+1$

<u>Subcase 2</u>

Assume p is associated with Q_j at stage σ

<u>Suspend</u> the association of p with Q_j and proceed as in case 1.

(A suspended association with a negative requirement can always be reinstated, but not through the same $\langle y,n\rangle$ of course!)

<u>Subcase 3</u>

p is now realized

If $[i](p) \neq 1$, add nothing to A or B and cancel all followers of R_e at stage σ, save for p, and all associations of such followers with negative requirements and go to stage $\sigma+1$.

If $[i](p) = 1$, proceed as in case 1.

Subcase 4

Define p to be σ. R_e is then said to receive attention through p at stage σ. Make p an unrealized follower of R_e.

Add nothing to A or B

Go to stage $\sigma+1$.

End of the construction

Remark:

Apparently a follower could be appointed and then (by Subcase 1) put in $A \cup B$ while prohibited. The following lemma proves that this cannot be the case.

Lemma 3.8

If R_e receives attention through p at stage σ, Then p is not prohibited after the beginning of stage σ (But p may become prohibited (again) at some later stage).

Proof

Case 1

If R_e requires attention through p, then as soon as stage σ begins, all prohibitions due to special requirements of lower priority are lifted. By definition of the functions p_0 and p_2 a positive requirement and a special requirement cannot have the same priority! So assume that p is prohibited because of some S_z of higher priority than R_e. But then there is a stage σ' such that p was in existence at stage σ' ($\therefore p < \sigma'$) and $\sigma' < \sigma$, such that p has been prohibited at stage σ'. As p was in existence at stage σ', it was already a follower of R_e (at stage σ'); and as R_e is of lower priority than S_z, p was cancelled at stage σ'. Thus R_e does not require attention through p at stage σ. $\Rightarrow\!\Leftarrow$

Case 2

p is appointed a follower of R_e at stage σ. But then p cannot be member of any prohibition established before stage σ. ⊣

Lemma 3.9

If p is a follower of R_e, then as soon as p is realized, either p enters $A \cup B$ or p is associated with some Q_z, or p enters α-A or α-B.

Proof

Assume p is created (as a follower) at stage σ. Thus $p = \sigma$, p is not prohibited (at stage σ). R_e receives attention at stage σ.

<u>If</u> p becomes realized at stage $\sigma' > \sigma$

<u>Then</u> R_e receives attention again and p is still in existence (at stage σ')

$\therefore$ p has not been prohibited for the sake of some S_j of higher priority than R_e (otherwise p would be cancelled) and p is no longer prohibited for the sake of some S_j of lower priority than that of R_e after the beginning of stage σ' (all these prohibitions have been lifted as soon as stage σ' began).

<u>If</u> $[i]_{\sigma'}(p) \neq 1$, <u>then</u> p enters α-A or p enters α-B. $\therefore$ p enters $\alpha\text{-}A \cup \alpha\text{-}B$

<u>If</u> $[i]_{\sigma'}(p) = 1$, <u>then</u> two cases can happen:

1) R_e has already as many realized followers as possible: then the $\langle y_0, n_0\rangle$ mentioned in subcase 1 of the construction is not well defined ($V_e^{\sigma'}(p) = \emptyset$ but if $p_1(z) = y$ and $y < p_0(e)$, then some q follower of R_e of higher order than p is already associated with Q_z $\therefore$ we cannot associate p with any Q_z)

$\therefore$ $\langle y_0, n_0\rangle$ is not well defined and p is not prohibited

$\therefore$ p enters $A \cup B$ without any problem.

2) Otherwise (i.e. $\langle y_0, n_0 \rangle$ is well defined): associate p with Q_{z_0} where z_0 is the unique z such that $p_1(z) = y_0$.

<u>Corollary 3.10</u>

<u>If</u> p is a follower of R_e, realized at stage σ, and if p is not associated <u>at the end of stage σ</u> with some Q_z,

<u>Then</u> the exact position of p is known at the end of stage σ

<u>Proof</u>

Assume R_e is $[i] \neq A$ (similarly for $[i] \neq B$)

$\therefore$ we know already that $p \notin B$

Then by lemma 3.9 we must either put p in A or in $\alpha - A$ at stage σ.

<u>Definition 3.16</u>

R_e is <u>discharged</u> (at stage σ), if R_e does not receive attention at stage σ', for any $\sigma' \geq \sigma$. R_e is <u>discharged by p</u> (at stage σ), if R_e does not receive attention through p at stage σ', for any $\sigma' \geq \sigma$.

S_e is <u>discharged</u> (at stage σ), if S_e does not receive attention at stage σ', for any $\sigma' \geq \sigma$.

Lemma 3.11

Each of the S_i's and R_i's is discharged.

Proof

1) Assume all R_i's and S_i's of higher priority than S_e have been discharged prior to stage σ. Assume S_e is special for B and x with $x \in \omega$ (there is obviously a similar argument for the case "S_e is special for A and x") and assume that S_e has not been discharged prior to stage σ.

$\therefore$ S_e must require attention at some stage $\sigma' \geq \sigma$.

$\therefore$ at stage σ' we have $\langle L_{\sigma'}(T), \varepsilon, T \upharpoonright \sigma', B^{\sigma'} \rangle \models \phi_x(0)$

As S_e is the requirement of highest priority that requires attention, S_e receives attention at stage σ'.

$\therefore$ we prohibit all ordinals $\beta < \gamma_0$ such that

$[B^{\gamma_0} = B^{\sigma'} \cap \gamma_0$ & $\langle L\gamma_0(T), \varepsilon, T \upharpoonright \gamma_0, B^{\gamma_0} \rangle \models \phi_x(0)$

& $\beta \notin B^{\gamma_0}$ & γ_0 is smallest such ordinal$]$

As all requirements of higher priority are discharged, this prohibition cannot be lifted at any stage σ'' such that $\sigma'' > \sigma'$

$\therefore$ S_e is discharged at stage σ'.

2) Assume all R_i's and S_i's of higher priority than R_e have been discharged prior to stage σ. Assume that R_e has not been discharged prior to stage σ.

2.1) Suppose that R_e is satisfied at some stage $\sigma' \geq \sigma$.

Say R_e is satisfied by p.

At a later stage p could become prohibited but not cancelled.

$\therefore$ p will still be in existence and R_e remains satisfied.

$\therefore$ R_e is discharged at stage σ'.

2.2) Suppose that R_e is not satisfied at any stage.

$\therefore$ each realized follower of R_e at stage $\sigma' \geq \sigma$ is associated with a different Q_u at the end of stage $\sigma' \geq \sigma$ [R_e is not satisfied at any stage, $\therefore$ p, a follower of R_e, cannot "enter" A, B, α-A or α-B unless it is cancelled, but as all requirements of higher priority are discharged prior to stage σ, no follower of R_e can be cancelled at later stages, $\therefore$ if p is a follower of R_e (at stage $\sigma' \geq \sigma$), as soon as it is realized, it must be associated with some Q_z (by lemma 3.9) and two different followers must be associated with different Q_z's (by construction)].

Consequently, R_e has at most $p_0(e)$ realized followers (at the end of stage σ') and at most one unrealized follower at stage σ'.

Let q_0 be the first follower of R_e of order 0 at _any_ stage σ' after stage σ. Then q_0 is never cancelled. By hypothesis R_e is never satisfied. If q_0 is always unrealized, then R_e is discharged (q_0 can only become prohibited, thus remains in existence!!!) If q_0 becomes realized, then (as R_e is never satisfied) q_0 is associated with some negative requirement (by lemma 3.9) and q_0 is associated with some Q_w (with $p_1(w) < p_0(e)$) for all sufficiently large stages. Let σ_0 be the stage at which q_0 is last associated with some "new" negative requirement (i.e. σ_0 is the first stage such that q_0 is associated with Q_w, say through $\langle y_0, n_0 \rangle$, and this association will never be cancelled).

$\therefore$ At stage σ_0+1 a follower q_1 of R_e is appointed, never to be cancelled.

And so on until termination with at worst $q_{p_0(e)+1}$. Either R_e is satisfied or some q_i is appointed to follow R_e and is never realized.

In either event R_e is discharged.

Lemma 4.12

A and B are subgeneric.

Proof

By lemma 3.7, as A and B are clearly Σ_1 over $\mathfrak{A}$, it is enough to show that both satisfy property H (definition 3.9).

$\therefore$ It is enough to show that each special requirement S_e is satisfied at some stage σ of the construction.

Assume S_e is special for B and i (similarly for A and i) and assume that S_e is not satisfied at any stage prior to stage σ. By lemma 3.11, there is a $\sigma' \geq \sigma$ such that S_e is discharged at stage σ'. By the proof of lemma 3.11, we know that

if there is a γ such that

$$\langle L\gamma(T), \varepsilon, T \upharpoonright \gamma, B^\gamma \rangle \models \phi_i(0)$$

then there is a γ such that $B^\gamma = B \cap \gamma$

and $\langle L\gamma(T), \varepsilon, T \upharpoonright \gamma, B^\gamma \rangle \models \phi_i(0)$

Remark:

Now that this fact is established, we feel more comfortable when we use definition 3.6.

Lemma 3.13

Neither A nor B is recursive.

Proof

Suppose $A = [e]$ for some $e \in \omega$

(The proof is similar for $B = [e]$)

If p is an unrealized follower of $[e] \neq A$, at stage σ, then p is either cancelled or eventually realized (If p remains eternally unrealized, then $[e](p)$ would diverge while $A(p) = 0$ or 1 according to $p \in A$ or $p \notin A$, $\therefore$ $A \neq [e]$ $\Rightarrow\!\Leftarrow$)

By lemma 3.11, $[e] \neq A$ is discharged

$\therefore$ there is a stage σ such that $[e] \neq A$ does not require attention at stage σ', for any $\sigma' \geq \sigma$.

$\therefore$ $[e] \neq A$ does not have an unrealized follower at stage σ (otherwise it would not be satisfied at stage σ, $\therefore$ never satisfied and either p would be cancelled or $[e]_{\sigma''}(p)$ would be defined for some $\sigma'' > \sigma$, and so $[e] \neq A$ would require attention after stage σ. $\Rightarrow\!\Leftarrow$)

Since $[e] = A$ has no unrealized follower at stage σ, and does not require attention at stage σ, $[e] \neq A$ must be satisfied prior to stage σ.

$\therefore$ there is a p that follows $[e] \neq A$ such that $p \in A \iff [e](p) = 1$

$\therefore$ $[e](p) \neq A(p)$ $\Rightarrow\!\!\!\Leftarrow$ $\dashv$

Lemma 3.14

Assume p follows R_i at stage σ

q follows R_j at stage σ

($\therefore$ p,q are both in existence at stage σ)

Assume furthermore that $p,q \in (A \cup B) - (A^\sigma \cup B^\sigma)$.

Let σ_1 be the first stage at which p enters $A \cup B$ and σ_2 be the first stage at which q enters $A \cup B$.

Assume $\sigma_1 < \sigma_2$

Then $p_0(i) > p_0(j)$.

Proof

As q is put in $A \cup B$ at stage $\sigma_2 > \sigma_1$, q is still in existence at stage σ_1 (it might be prohibited but not cancelled, because a cancelled follower can never be reappointed) and q is not cancelled at stage σ_1 (q might be prohibited at stage σ_1)

∴ R_j must have higher priority than R_i.

Reformulation

If p and q are both follower "in existence" at some "early" stage, and if $p \in A \cup B$, say p enters A B at stage σ, and if q is still in existence, but has not yet been put in $A \cup B$, at the end of stage σ (it might be the case that q will never enter $A \cup B$).

Then p has been appointed follower for the sake of a requirement of lower priority than the priority of the positive requirement for which q was appointed.

Lemma 3.15

$(A \cap \sigma) - A^{\sigma}$ and $(B \cap \sigma) - B^{\sigma}$ are finite.

Proof

(We give a proof for A, the proof for B is similar)
We want to show that $(A \cap \sigma) - A^{\sigma}$ is finite
(i.e.: there are only finitely members of A, appointed before stage σ which enters A after stage σ)
Fix $\sigma < \alpha$
If $z \in A \cap \sigma$, then $z \in A^{\sigma}$ or z is a follower in existence at stage σ (because $z < \sigma$)

Let $\sigma_0 \geq \sigma$ be the least stage such that some $z < \sigma$ is put in A at stage σ_0 (say it is z_0)

For each $i \in \omega$, let σ_{i+1} be the least stage τ, $\tau \geq \sigma_i$ such that some $z < \sigma$ enters A at stage σ_{i+1} (say $z = z_{i+1}$)

Suppose σ_i is well defined for all $i \in \omega$

Let R_{k_i} be the requirement satisfied at stage σ_i by z_i.

As all these z_i's were existing followers at the beginning of stage σ, by lemma 3.14 we get:

$p_0(k_0) > p_0(k_1) > p_0(k_2) > \ldots \qquad \Rightarrow\!\Leftarrow$

If σ_0 is not defined then $z \in A \cap \sigma \iff z \in A^{\sigma} \cap \sigma$

If σ_n is the last well defined σ_i ($n \in \omega$), then $z \in A \cap \sigma \iff z \in A^{\sigma_n} \cap \sigma$.

It is now obvious that A and B are regular.

<u>Lemma 3.16</u>

<u>If</u> $C = [(i)_0]^A = [(i)_1]^B$

<u>Then</u> C is recursive

Proof

Let σ_1 be the least ordinal σ such that R_y has been discharged prior to stage σ for every y such that $p_0 y < p_1 i = 5i$. The existence of σ_1 follows from the proof of lemma 3.11.

Any positive requirement that receives attention at stage β, $\beta \geq \sigma_1$, has its followers subject to association with Q_i at stage β.

To decide whether or not $x \in C$, search for a stage $\sigma_2 \geq \sigma_1$ such that $L(\sigma_2,i) = M(\sigma_2,i) > x$. σ_2 exists by lemma 3.15.

$$\therefore \quad [(i)_0]_{\sigma_2}^{A^{\sigma_2}}(x) = [(i)_1]_{\sigma_2}^{B^{\sigma_2}}(x) = q_1 \quad \text{for some} \quad q_1 \in \alpha .$$

A and B are regular and subgeneric,

$$\therefore \left[\begin{array}{l} [(i)_0]^A(x) = \lim\limits_{\sigma \to \alpha} [(i)_0]_\sigma^{A^\sigma}(x) \\ \\ [(i)_1]^B(x) = \lim\limits_{\sigma \to \alpha} [(i)_1]_\sigma^{B^\sigma}(x) \end{array}\right.$$

$\therefore$ to show that $C(x) = \delta$ it is enough to show that

$$([(i)_0]_\tau^{A^\tau}(x) = \delta \quad \text{or} \quad [(i)_1]_\tau^{B^\tau}(x) = \delta) \quad \text{for all} \quad \tau \geq \sigma_2$$

Part I

At stage σ_2, we have:

$$[(i)_0]^{A^{\sigma_2}}_{\sigma_2}(x) = [(i)_1]^{B^{\sigma_2}}_{\sigma_2}(x) = q_1$$

$$\therefore \quad (J) \left[\begin{array}{l} \langle L_{\sigma_2}(T), \varepsilon, T \upharpoonright \sigma_2, A^{\sigma_2} \rangle \models \phi_{(i)_0}(x, q_1) \\ \\ \langle L_{\sigma_2}(T), \varepsilon, T \upharpoonright \sigma_2, B^{\sigma_2} \rangle \models \phi_{(i)_1}(x, q_1) \end{array} \right.$$

where $\phi_{(i)_0}$ and $\phi_{(i)_1}$ are the Σ_1 formulas used to code the two functions we are considering.

As A and B are subgeneric, we can consider without problem the objects c_0 and d_0, where c_0 (d_0 respectively) is the computation of

$$[(i)_0]^{A^{\sigma_2}}_{\sigma_2}(x) = q_1 \quad ([(i)_1]^{B^{\sigma_2}}_{\sigma_2}(x) = q_1)$$

c_0 (d_0 respectively) will be invalid at stage $\beta > \sigma_2$ only if some $z < \sigma_2$ is put in A(B) before β but after σ_2 (This is obvious from (J)). Let β_1 be the least $\beta \geq \sigma_2$ such that some $z < \sigma_2$ is put in A or B at stage β. Let z_1 be a z put in A at stage β_1, let z_1 follow R_{y_1} ($\therefore p_0 y_1 = 5y_1 + 1$)

The computation d_0 is still valid at the beginning of stage β_1+1,

<u>Claim 1</u>

<u>If</u> p is a follower of R_v such that

1) p is in existence at the end of stage β_1

2) R_v is not satisfied before the end of stage β_1

<u>Then</u> $p < \sigma_2$

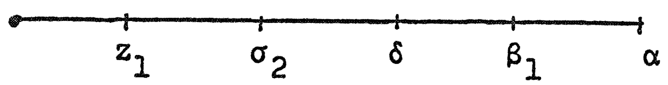
<u>Proof of the claim</u>

Let R_v be a requirement such that R_v has a follower p in existence at the end of stage β_1 and R_v is not satisfied before the end of stage β_1.

$\therefore$ $p_0(v) < p_0(y_1)$ (B)

Assume now that $p \geq \sigma_2$.

$\therefore$ there is a δ such that $\sigma_2 \leq \delta < \beta_1$ and p is appointed to follow R_v at stage δ. Since $z_1 < \sigma_2$, z_1 has been appointed before stage σ_2; as z_1 will enter A at stage β_1, z_1 is not cancelled at stage δ.



$\therefore$ $p_0 v > p_0 y_1$ because z_1 is not cancelled because of the appearing p.

But this contradicts (B).

$\therefore$ all followers in existence at the end of stage β_1, for the sake of requirements which are not satisfied before the end of stage β_1 belong to σ_2

claim 1

Part II

If no $z < \sigma_2$ is put in B after β_1,

then the computation d_0 is valid forever and

$[(i)_1]_\beta^{B^\beta}(x) = q_1$ for all $\beta \geq \sigma_2$

$\therefore$ q_1 is the true value of $[(i)_1]^B(x)$

So assume that there is a $\beta > \beta_1$, such that some $z < \sigma_2$ is put in B^β at stage β. Let the least such β be β_2, and let z_2 be such a z

Suppose z_2 follows R_{y_2}

We must notice that z_2 will enter B at stage β_2. Thus z_2 will not be cancelled at any stage $\gamma \leq \beta_2$. As z_2 will enter B at stage β_2, z_2 must be realized at some stage $\gamma_1 \leq \beta_2$; we know also that R_{y_2} will receive attention at least once "between" stage β_1 and stage β_2 (namely at stage β_2)

Two cases occur:

Case 1

z_2 is not associated with Q_i at any stage β such that $\beta_1 \leq \beta \leq \beta_2$. By the preceding remarks we know that R_{y_2} will receive attention through z_2 at some stage β such that $\beta_1 < \beta \leq \beta_2$. Let β' be such a β (the first one for instance)

Since $z_2 < \sigma_2 < \beta_1$, z_2 existed as a follower at the end of stage β_1.

If z_2 was unrealized at the end of stage β_1, then z_2 was eligible (at stage β') for association with Q_i through $<5i,0>$; and if z_2 was associated with some Q_j through, say $<w,n>$, at the end of stage β_1, then z_2 was eligible (at stage β') for association with Q_i through $<5i,n+1>$.

But since z_2 is not associated with Q_i at stage β' we must have:

"some follower r of R_{y_2} of higher order than that of z_2 was associated with Q_i at stage ρ, where ρ is the least stage such that z_2 was associated with some Q_j through some $<u,k>$" (if there is no such ρ, let $\rho = \beta_2$).

We have $\langle u,k\rangle > \langle 5i,0\rangle$ if z_2 was unrealized at stage β_1, and $\langle u,k\rangle > \langle 5i,n+1\rangle$ if z_2 was realized at stage β_1 and associated then with some Q_j through $\langle w,n\rangle$.

But since r has higher order than z_2 at stage ρ, $r < z_2$; so r existed as a follower at stage σ_2. If r were not associated with Q_i at stage σ_2, then it must be the case that R_{y_2} has received attention through r at stage δ ($\sigma_2 \le \delta < \beta'$) because r was not associated with Q_i at stage σ_2 but will be so at stage $\beta' \le \beta_2$. But z_2 would have been cancelled at stage δ and thus could not be a follower at stage β_2. $\Rightarrow\Leftarrow$

$\therefore$ r is associated with Q_i at stage σ_2.

But σ_2 is such that $L(\sigma_2,i) = M(\sigma_2,i)$, i.e. Q_i is satisfied by σ_2.

$\therefore$ r is a follower of R_{y_2} which is associated to a negative requirement at stage σ_2, and R_{y_2} is not satisfied prior to stage σ_2.

$\therefore$ R_{y_2} requires attention through r at stage σ_2

There are now four possibilities:

1) R_{y_2} receives attention through r and z_2 is cancelled $\Rightarrow\!\Leftarrow$

2) R_{y_2} receives attention through a follower of higher order than r and z_2 is cancelled $\Rightarrow\!\Leftarrow$

3) R_v receives attention, with $p_0 v < p_0 y_2$ and z_2 is cancelled $\Rightarrow\!\Leftarrow$

4) S_v receives attention with $p_2 v < p_0 y_2$ and z_2 is cancelled $\Rightarrow\!\Leftarrow$

Thus case 1 cannot occur.

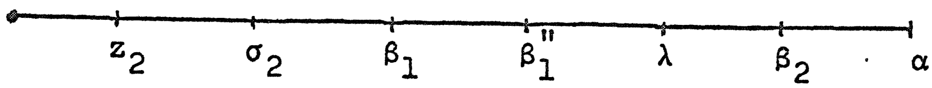
Case 2

z_2 is associated with Q_i at stage β for some β such that $\beta_1 \leq \beta \leq \beta_2$; let β_1' be the least such β.

Since z_2 enters B at stage β_2, there is a first stage β_1'' such that $\beta_1' < \beta_1'' \leq \beta_2$ and the association of z_2 with Q_i is cancelled at stage β_1''. Since z_2 is still a follower at the end of stage β_1'', it must be the case that $L(\beta_1'',i) = M(\beta_1'',i)$ (otherwise Q_i is not satisfied by β_1'', $\therefore$ R_{y_2} does not receive attention through z_2, $\therefore$ we cannot cancel the association of z_2 with Q_i unless we cancel z_2 $\Rightarrow\!\Leftarrow$) and there must be an ordinal $q_2 < \alpha$ and a computation c_1 of

$$[(i)_0]^{A^{\beta_1''}}_{\beta_1''}(x) = q_2 \quad (\text{because } L(\beta_1'',i) = M(\beta_1'',i) > x$$

$$\Rightarrow [(i)_0]^{A^{\beta_1''}}_{\beta_1''}(x)\downarrow)$$



If c_1 were invalid at the end of stage β_2, then some follower r would land in A at some stage λ $(\beta_1'' < \lambda < \beta_2)$

Suppose r follows R_v.

r has to be such that $r < \beta_1''$ (otherwise it would not invalidate c_1) and r exists as a follower at stage β_1''. Also $p_0 v < p_0 y_2$ because R_{y_2} has a follower z_2 which exists at stage β_1'', and R_{y_2} requires attention through z_2 at stage β_1'' but does not cancel r (which will enter A after stage β_1'' and has been appointed a follower before stage β_1'').

But as R_v receives attention at stage λ, z_2 would be cancelled at stage $\lambda < \beta_2$. $\Rightarrow\!\Leftarrow$

Hence c_1 is valid at the end of stage β_2.

But all followers in existence at the end of stage β_2 for the sake of requirements which are not satisfied before the end of stage β are less than σ_2 (this can be proven as claim 1, in part I, with z_2 for z_1, β_2 for β_1 and B for A).

Let $c_2 = c_1$

If no $z < \sigma_2$ is placed in A after stage β_2, then the computation c_2 is valid forever

Assume that some $z < \sigma_2$ is placed in A after stage β_2.

Continue to alternate as above between A and B. If for some $n \in \omega$, z_n fails to be defined, then the lemma is proved. So suppose z_n is defined for all $n \in \omega$. As z_n follows R_{y_n} and is in existence at stage σ_2, we get by lemma 3.14

$$p_0(y_1) > p_0(y_2) > p_0(y_3) > \ldots \qquad \Rightarrow\!\Leftarrow$$

$\dashv$

We might, of course, wish to use the full power of the definitions of "F-calculable in" and of "subgeneric" given by Harrington in [8]. But in that case, we need more than countably many requirements (as soon as $n > 1$).

Lerman and Sacks called <u>refractory</u> the Σ_1-admissible ordinals σ such that, in σ-recursion theory (i.e. in $\langle L_\sigma, \varepsilon\rangle$), the following holds: $p2\sigma = gc\sigma < tp2\sigma \leq \sigma$ (with: $p2\sigma = \Sigma_2$ projectum (in L_σ) of σ; $gc\sigma$ = greatest cardinal in L_σ, if there is one, or σ otherwise; $tp2\sigma$ = tame Σ_2 projectum (in L_σ) of σ). They proved in [8], that if σ is <u>not</u> refractory, then there is a minimal pair in L_σ. Unluckily their construction might lead to two σ-r.e. subsets of σ whose indices are not integers: this cannot lead to a minimal pair of sets of individuals semirecursive in F, but well to objects semirecursive in $\langle F, f\rangle$ for some subindividual f. Nevertheless we can use a technique derived from that used by Shore in [27] for his uniform solution of Post's problem.

<u>Definition 3.17</u>

An ordinal σ is said to be <u>distorted</u> iff

$p2\sigma \leq gc\sigma < tp2\sigma \leq \sigma$

<u>Theorem 3.17</u>

<u>If</u> $p2\sigma = \sigma$

<u>Then</u> there is a minimal pair of σ-r.e. sets A and B such that both A and B have integers as indices.

The proof requires first a Shore-like uniform construction and an argument which follows, after translation into the corresponding language, the Lerman-Sacks argument of [8].

<u>Corollary 3.18</u>

<u>If</u> $p2\alpha = \alpha$

<u>Then</u> there are two (uncountable if $n > 1$) objects of type $n+2$, A and B, semirecursive in F, such that every object of type $n+1$, D, corresponding to a set of ordinals less than α which is reducible to both A and B <u>according to Harrington's definition</u> is recursive in F. Moreover A is not reducible to B (nor B to A).

The proof uses the uniform construction of theorem 3.17 and the argument used by Harrington in [8] to show the existence of type $n+2$ objects which solve Post's problem.

<u>Remark 3.19</u>

As announced earlier, if $V = L$, then λ^F_{n-1} is not distorted.

We must confess that we do not know any model of ZFC where λ^F_{n-1} is distorted; also: although there are distorted ordinals (e.g. the first Σ_2-admissible), we do not know whether λ^F_{n-1}, if distorted, could be non-refractory (the first Σ_2-admissible is refractory).

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Biography

Francis D. Lowenthal was born in Brussels, Belgium, on June 2, 1947. His first act, when he became a student at the University of Brussels, was to become member of "Le Cercle du Libre Examen". After May 1968, he became active in numerous commissions and subcommissions trying to reform the structures of the university.

He received his "Licence en Sciences Mathematiques" with "la plus grande distinction" from the University of Brussels, in 1970. He became then assistant at the university and received in 1972 his "Getuigschrift van Grondige Studies in de Wiskunde". He left then Brussels for Cambridge, Mass. with a Francqui - B.A.E.F. (C.R.B.) fellowship and became Hallam Tuck fellow 1973. He was supported during his last year of graduate studies by an M.I.T. teaching assistantship.

He learned many things in Cambridge, some even connected with mathematical logic. Some people might say that he learned to read and write in America.